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A geometric quadratic form of 3-dimensional normal maps

Masaharu Morimoto [1]

Department of Environmental and Mathematical Sciences, Faculty of Environmental Science and Technology, Okayama University, Tsushimanaka, Okayama, 700 Japan

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Abstract

Associated to a homological surgery problem (f, b) consisting of a degree 1 map $f: X \to Y$ between compact, oriented, 3-dimensional manifolds and a stable vector bundle map $b: T(X) \to \eta$ covering f, we obtain a butterfly diagram of homology kernels, and a quadratic module $(K_1(\partial U; \mathbb{Z}), q)$, where $q: K_1(\partial U; \mathbb{Z}) \to \mathbb{Z}/2\mathbb{Z}$. The quadratic form q is related to a map $\overline{\mu}_{\kappa}$ from a set of immersions $S^1 \times D^2 \to X$ to $\mathbb{Z}/2\mathbb{Z}$. The $\overline{\mu}_{\kappa}$ is defined in connection with frames of tangent bundles. Using the geometric interpretation $\overline{\mu}_{\kappa}$ of q, we prove that $K_2(X, U; \mathbb{Z})$ is a Lagrangian of $(K_1(\partial U; \mathbb{Z}), q)$. © 1998 Elsevier Science B.V.

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1. Introduction

This paper consists of results only in 3-dimensional topology, although the motivation arose in transformation groups.

The ordinary surgery theory by Wall [15] on compact manifolds of dimension $n \geqslant 5$ is an obstruction theory to convert a degree 1 map $f: X \to Y$ to a homotopy (or simple homotopy) equivalence by surgery on X. We can make an equivariant analogue assuming the gap hypotheses (cf. [2,3,6,8,11,13]). If G is a finite group, by Smith's theorem [4, p. 130, Theorem 5.2], the procedure that converts a G-map $f: X \to Y$ to a homotopy equivalence necessarily includes the subprocedure that converts $f^P: X^P \to Y^P$ to a mod p homology equivalence for each p-subgroup P of G (p is a prime). Even if $\dim X \geqslant 5$, the dimension of X^P may be low, i.e., $\leqslant 4$. Thus in some applications of

[1] E-mail: morimoto@math.ems.okayama-u.ac.jp.

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surgery theory on manifolds of dimension $\geqslant 5$, it occurs that homological surgery theory on low-dimensional manifolds are also required. Since our surgery in applications needs bundle data, our low-dimensional, homological surgery theory should be constructed in the category with bundle data.

Let us explain it with examples. Up to 1989, the existence of smooth, A_5-actions on spheres S^n such that the fixed point set consists of a single point were known for each $n = 6, 10$ and $n \geqslant 12$ [9,10], where A_5 is the alternating group of degree 5. In 1990, Buchdahl, Kwasik and Schultz [5, Theorem II.5] made locally linear, A_5-actions on S^n with exactly one fixed point for all $n \geqslant 6$. Thus we conjectured the existence of smooth, one fixed point actions on S^n for all $n \geqslant 6$. Their method in [5] has its roots in the locally linear category and is hardly extensible to a method in the smooth category. The cases where $n = 6$ and $n \geqslant 9$ were affirmatively proved in [12] (1991) and the case where $n = 7$ was done in [1] (1992), while the case where $n = 8$ was left as a delicate problem to answer. Recently we could answer affirmatively to the case where $n = 8$, too. A theory of homological surgery on 3-dimensional manifolds is available to construct smooth, one fixed point A_5-actions on S^n for $n = 6, 7, 8$ and 9 as follows.

First we explain the case where $n = 7$ (respectively 9). Let V be a real A_5-module of dimension 7 (respectively 9) obtained as the direct sum of irreducible modules of dimension 3 and 4 (respectively of dimension 3). Using the transversality construction, one can obtain an A_5-normal map $f : X \to Y = S(\mathbb{R} \oplus V)$ such that $|X^{A_5}| = 1$. As a step previous to the step where we obtain a homotopy equivalence $f' : X' \to Y$ by A_5-surgery on X such that $X'^{A_5} = X^{A_5}$, we have to modify f so that $f^{C_p} : X^{C_p} \to Y^{C_p}$ are mod p homology equivalences for $p = 2, 3$. Since $\dim X^{C_p} = 3$, we can use 3-dimensional surgery theory here.

Next we explain the case where $n = 8$ (respectively 6). Let V be a real A_5-module of dimension 8 (respectively 6) obtained as the direct sum of irreducible modules of dimension 3 and 5 (respectively of dimension 3). We can obtain not only an A_5-normal map $f : X \to Y = S(\mathbb{R} \oplus V)$ such that $|X^{A_5}| = 1$ but also H-normal cobordisms $F_H : W_H \to Y \times [0, 1]$ between $\operatorname{Res}_H f$ and $\operatorname{Res}_H \mathrm{id}_Y$ for all proper subgroups H of A_5. After modifying f so that $f^H : X^H \to Y^H$ are homology equivalences for all subgroups $H \neq \{1\}, A_5$, we reach the step where we like to evaluate the obstruction $\sigma(f)$ to converting f to a homotopy equivalence by A_5-surgery on points in X of isotropy subgroup $\{1\}$. For this evaluation, we use an A_4-normal cobordism

$$F_{A_4} : W_{A_4} \to Y \times [0, 1]$$

such that $F_{A_4}^{C_3} : W_{A_4}^{C_3} \to Y^{C_3} \times [0, 1]$ is a homology equivalence. Since $\dim W_{A_4}^{C_3} = 3$, we can use 3-dimensional surgery theory to obtain such a nice F_{A_4} and finally show $\sigma(f) = 0$. The details will appear elsewhere.

We desire to obtain a homological surgery theory on 3-dimensional manifolds with bundle data. But previous to the surgery theory, we would like to establish a fact (i.e., Theorem 8.1) and this is the purpose of the present paper. The fact is concerned with a quadratic form on the surgery kernel of a normal map and is a key to the 3-dimensional, homological surgery theory. The completion with Theorem 8.1 to the 3-dimensional surgery theory will be discussed in another paper.

In order to see why Theorem 8.1 is needed to our surgery theory, let $f : X \to Y$ be a degree 1 map from a compact, oriented, 3-dimensional manifold to another. Suppose that one could obtain the butterfly diagram with coefficient ring $\mathbb{Z}$, associated to f:

$$
\begin{tikzcd}
0 \arrow[dr] \arrow[rrrr, bend left] & & K_{k+1}(X, X_0) \arrow[rrrr, bend left] & = & K_{k+1}(U, \partial U) \arrow[dr] & & K_k(X_0) \arrow[dr] \arrow[rr, bend left] & & 0 \\
& K_{k+1}(X) \arrow[ur] \arrow[dr] & & & & K_k(\partial U) \arrow[ur] \arrow[dr] & & K_k(X) \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] \arrow[rrrr, bend right] & & K_{k+1}(X, U) \arrow[rrrr, bend right] & = & K_{k+1}(X_0, \partial U) \arrow[ur] & & K_k(U) \arrow[ur] \arrow[rr, bend right] & & 0
\end{tikzcd}
$$

where $k = 1$, similarly to [15, p. 56] or [11, Diagram 4.2]. Let $\{e_1, \ldots, e_m, f_1, \ldots, f_m\}$ be the canonical $\mathbb{Z}$-basis of $K_k(\partial U)$. The quadratic form $q : K_k(\partial U) \to \mathbb{Z}/2\mathbb{Z}$ is defined by

$$q\Big(\sum_i (a_i e_i + b_i f_i)\Big) = \sum_i [a_i b_i] \quad (a_i, b_i \in \mathbb{Z}).$$

The ordinary surgery obstruction of dimension $4h + 3 \geqslant 5$ (hence $k = 2h + 1$) is a certain equivalence class of a quadratic formation (i.e., a quintuple like

$$\big(K_k(\partial U), B, q, K_{k+1}(U, \partial U), K_{k+1}(X_0, \partial U)\big),$$

where $B : K_k(\partial U) \times K_k(\partial U) \to \mathbb{Z}$ is the intersection form, although this is not exact if the fundamental group is nontrivial). A significant point to showing that the equivalence class becomes the surgery obstruction is that $L_1 = K_{k+1}(U, \partial U)$ and $L_2 = K_{k+1}(X_0, \partial U)$ are Lagrangians, namely $B(L_i, L_i) = 0$ and $q(L_i) = 0$. It is easy to show that $B(L_1, L_1) = 0$, $q(L_1) = 0$ and $B(L_2, L_2) = 0$. The property that $q(L_2) = 0$ was proved with a geometric interpretation of q, namely the self-intersection form μ. An element $z \in K_{k+1}(X_0, \partial U)$ is represented by the diagram

$$
\begin{tikzcd}
S^k \arrow[r, "\phi"] \arrow[d] & \partial U \arrow[d] \\
D^{k+1} \arrow[r, "\Phi"] & X_0
\end{tikzcd}
.
$$

If we like to regard z as an element in $K_k(\partial U)$ then it is the element represented by $\phi : S^k \to \partial U$. Using the diagram, we can show that the self-intersection number $\mu(z)$ is trivial, and consequently $q(z)$ is trivial. But the diagram is not straightforwardly available in the case where $k = 1$ because the Hurewicz homomorphisms are not necessarily isomorphisms. Thus we must find a substitute to μ. Moreover, our surgery in the application [2, Theorem 1.1] is performed on equivariant maps together with bundle data. That is, we have to guarantee that the surgery associated to certain elements x of $K_1(\partial U)$ such that $B(x,x)=0$ and $q(x)=0$ necessarily extends the bundle data. Hence the value of the substitute must be decided in connection with bundle data. The main result of the current paper, namely Theorem 8.1, guarantees that the map $\overline{\mu}$ defined in Section 2 and used in the context of Section 7 is a desired substitute. That is, using $\overline{\mu}$, we shall prove that $K_2(X,U)$ is a Lagrangian of $K_1(\partial U)$ in a reasonable situation.

Our $\overline{\mu}$ will be obtained from the following idea. Let $f:X\to Y$ be a degree 1 map from a compact, oriented, 3-dimensional, smooth manifold to another, η a real vector bundle over Y, and $b:T(X)\to\eta$ a vector bundle map covering f. Let y_0 be a point in Y. For a smooth embedding $\alpha:S^1\times D^2\to X$ such that $f(\mathrm{Im}(\alpha))=y_0$, the value $\overline{\mu}(\alpha)\in\mathbb{Z}/2\mathbb{Z}$ is defined to be 0 if and only if the bundle data b is extensible to bundle data over the trace of surgery on f along α. The relation between q above and $\overline{\mu}$ will be established in Theorem 6.3.

Our main theorem (Theorem 8.1) will be proved according to the following idea. Let $z\in K_2(X_0,\partial U)$. Then z can be realized as the boundary of an immersed surface $(A,\partial A)\looparrowright(X_0,\partial U)$ by virtue of the Poincaré–Lefschetz duality, Steenrod's obstruction theory and the general position argument. On the other hand, we associate to the z regarded as an element in $K_1(\partial U)$, a smooth embedding $\alpha:S^1\times D^2\to\mathrm{Int}(U)$ such that $\alpha(S^1\times 1)$ is cobordant in U to ∂A. Using Steenrod's obstruction theory, we show that the bundle data over X extends to bundle data over the trace of surgery along α if the bundle data around ∂A extends to bundle data over A. The last is always satisfied and hence $\overline{\mu}(\alpha)=0$. By Theorem 6.3, we conclude $q(z)=0$.

The remainder of the paper is organized as follows. In Section 2, for a 3-dimensional, smooth manifold V and a frame κ over V, we define a $\mathbb{Z}/2\mathbb{Z}$-valued function $\overline{\mu}_\kappa$ on a set of smooth immersions $\alpha:S^1\times D^2\to V$. In Section 3, we prove that $\overline{\mu}_\kappa(\alpha)$ vanishes if and only if a frame over $S^1\times D^2$ canonically obtained from κ by $\mathrm{d}\alpha$ is extensible to a frame over $D^2\times D^2$ (Theorem 3.4). Section 4 is devoted to elementary notion in surgery theory. In Section 5, we show that $\overline{\mu}_\kappa(\alpha)$ is decided by the homology element given by the longitude of α. If V is the solid torus of genus m, then we obtain a simple formula of $\overline{\mu}_\kappa(\alpha)$ (Theorem 6.3 and Definition 6.1). In Section 7, we define a normal map (f,b) consisting of a map $f:X\to Y$ and a stable vector bundle map $b:T(X)\to\eta$ covering f. A normal map determines a butterfly diagram as above and a frame κ over U. Then the value $\overline{\mu}_\kappa(\alpha)$ is an invariant of the cobordism class of α. This is established as Theorem 7.5. We prove the main result in Section 8 using Theorem 7.5.

Notation.

$\mathbb{Z}$ is the ring of integers;
$\mathbb{R}$ is the real number field;
$\mathbb{C}$ is the complex number field;
I is the unit interval $[0,1]$ in $\mathbb{R}$;
S^{n-1} is the unit sphere in n-dimensional Euclidean space $\mathbb{R}^n$;

D^n is the closed unit disk in $\mathbb{R}^n$;
$[a..b]$ is the set of integers s such that $a \leqslant s \leqslant b$.

2. Definition of $\overline{\mu}$

In the current paper, each manifold will be a paracompact, smooth manifold possibly with boundary unless specifically mentioned. Let M and N be oriented manifolds. We regard the cartesian product $M \times N$ as a smooth manifold by smoothing corners if necessary. Let $[M]$ and $[N]$ denote the orientations of M and N, respectively. So $[M]$ assigns a basis (more precisely a class of bases) $[M]_x$ of the tangent space $T_x(M)$ to each point $x \in M$. Define the orientation $[M \times N]$ of $M \times N$ by assigning to each $(x, y) \in M \times N$ the basis $[M]_x$ followed by $[N]_y$ of $T_{(x,y)}(M \times N)$ (i.e., if $[M]_x = (e_1, \dots, e_m)$ $(e_i \in T_x(M))$ and $[N]_x = (f_1, \dots, f_n)$ $(f_i \in T_y(N))$ then $[M \times N]_{(x,y)} = ((e_1, 0), \dots, (e_m, 0), (0, f_1), \dots, (0, f_n))$ under the canonical identification $T_{(x,y)}(M \times N) = T_x(M) \oplus T_y(N)$). If the boundary ∂M is not empty, then let $\nu(\partial M, M)$ denote the orthogonal complement (with respect to some Riemannian metric on M) of the tangent bundle $T(\partial M)$ of ∂M in $T(M)_{\partial M}$ and $\boldsymbol{\nu} = \boldsymbol{\nu}_{\partial M} : \partial M \to \nu(\partial M, M)$ denote the section assigning to $x \in \partial M$ the outward normal vector of length 1, in $\nu_x(\partial M, M)$. We call $\boldsymbol{\nu}$ the *normal section* of ∂M. The boundary ∂M has the orientation $[\partial M]$ such that $\boldsymbol{\nu}(x)$ followed by $[\partial M]_x$ determines the same orientation as $[M]_x$ for each $x \in \partial M$. Let $\mathbb{C}$ (respectively $\mathbb{R}$) denote the complex (respectively real) number field and S^1 (respectively D^2) the unit sphere (respectively the closed unit disk) of $\mathbb{C}$. These manifolds have the standard orientations.

The inclusion map $S^1 \to S^1 \times D^2$; $x \mapsto (x, 1)$ (respectively $x \mapsto (1, x)$) is called the *longitude* (respectively *meridian*) of $S^1 \times D^2$. The closed unit interval I has the standard orientation.

Let X be a 3-dimensional manifold and ρ an oriented, real vector bundle over X with fiber $\mathbb{R}^n$. Then ρ determines the associated oriented n-frame bundle $P^+(\rho)$ over X with fiber $\cong GL_n^+(\mathbb{R})$. For $A \subseteq X$, we call a cross-section $A \to P^+(\rho)_A$ a *frame over A of ρ*. If $\boldsymbol{\rho}_i$ $(i = 1, 2)$ are frames over A of vector bundles ρ_i, then the frame $\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2$ over A of $\rho_1 \oplus \rho_2$ is defined by

$$(\boldsymbol{\rho}_1 + \boldsymbol{\rho}_2)(x) = \left(\left(e_1^{(1)}(x), 0\right), \dots, \left(e_{m(1)}^{(1)}(x), 0\right), \left(0, e_1^{(2)}(x)\right), \dots, \left(0, e_{m(2)}^{(2)}(x)\right)\right).$$

where $x \in A$ and $\boldsymbol{\rho}_i(x) = (e_1^{(i)}(x), \dots, e_{m(i)}^{(i)}(x))$ $(i = 1, 2)$. Let $\varepsilon_X(\mathbb{R}^m)$ denote the trivial bundle over X with fiber $\mathbb{R}^m$ and $\boldsymbol{\varepsilon}_{X,m}$ the canonical frame over X of $\varepsilon_X(\mathbb{R}^m)$ (i.e., if $(e_1, \dots, e_m)$ is the canonical basis of $\mathbb{R}^m$, then $\boldsymbol{\varepsilon}_{X,m}(x) = (x, (e_1, \dots. e_m))$ for $x \in X$). A *stable frame* over A of ρ means a frame over A of $\rho \oplus \varepsilon_X(\mathbb{R}^m)$ for some integer $m \geqslant 0$. If $\boldsymbol{\rho}_A : A \to \rho$ is a frame over A of ρ, then the stable frame $\boldsymbol{\rho}_A$ over A of ρ means the frame $\boldsymbol{\rho}_A + \boldsymbol{\varepsilon}_{A,m}$ over A of $\rho \oplus \varepsilon_X(\mathbb{R}^m)$.

Since S^1 is a Lie group, from the unit vector u with positive orientation in the tangent space $T_1(S^1)$ and left translations $L_g : S^1 \to S^1$; $x \mapsto gx$ $(g \in S^1)$, we obtain the left S^1-invariant frame $\boldsymbol{\tau}_\ell : S^1 \to T(S^1)$; $g \mapsto \mathrm{d}L_g(u)$, where $\mathrm{d}L_g : T(S^1) \to T(S^1)$ is the differential of L_g. Let $\boldsymbol{\tau}_s$ be the frame over D^2 of $T(D^2)$, obtained from $\boldsymbol{\varepsilon}_{D^2,2}$ via the canonical identification $T(D^2) = \varepsilon_{D^2}(\mathbb{R}^2)$. We get the frame $\boldsymbol{\tau}_\ell \times \boldsymbol{\tau}_s$ over $S^1 \times D^2$ of $T(S^1 \times D^2) = T(S^1) \times T(D^2)$. Define the frame $\boldsymbol{\tau}_p$ over $S^1 \times D^2$ of $T(S^1 \times D^2)$ by

$$\boldsymbol{\tau}_p(g, y) = \left(\boldsymbol{\tau}_\ell(g), \mathrm{d}L_g\left(\boldsymbol{\tau}_s\left(g^{-1}y\right)\right)\right) \quad \left(g \in S^1,\ y \in D^2\right).$$

Let $\boldsymbol{\nu}_{S^1} : S^1 \to \nu(S^1, D^2)$ be the normal section. Using the identification

$$T\left(D^2 \times D^2\right)_{S^1 \times D^2} = T\left(D^2\right)_{S^1} \times T\left(D^2\right) = \left(\nu(S^1, D^2) \oplus T(S^1)\right) \times T\left(D^2\right),$$

define a section $\boldsymbol{\nu}_{S^1 \times D^2} : S^1 \times D^2 \to T(D^2 \times D^2)_{S^1 \times D^2}$ by

$$\boldsymbol{\nu}_{S^1 \times D^2}(x, y) = \left(\left(\boldsymbol{\nu}_{S^1}(x), 0_x\right), 0_y\right) \quad \left((x, y) \in S^1 \times D^2\right)$$

where 0_x and 0_y are zero vectors in $T_x(S^1)$ and $T_y(D^2)$, respectively.

If $\boldsymbol{\tau}$ is a frame over $S^1 \times D^2$ of $T(D^2 \times D^2)$, then using the canonical identifications $S^1 \times \{0\} = S^1$ and $T(D^2 \times D^2)_{S^1 \times \{0\}} = T(D^2)_{S^1} \oplus \varepsilon_{S^1}(\mathbb{R}^2)$, $\boldsymbol{\tau}|_{S^1 \times \{0\}}$ is regarded as a frame over S^1 of $T(D^2)_{S^1} \oplus \varepsilon_{S^1}(\mathbb{R}^2)$. We note that $\boldsymbol{\tau}$ is extensible to a frame over $D^2 \times D^2$ of $T(D^2 \times D^2)$ if and only if $\boldsymbol{\tau}|_{S^1 \times \{0\}}$ is extensible to a frame over D^2 of $T(D^2) \oplus \varepsilon_{S^1}(\mathbb{R}^2)$. The next proposition follows from the fact that $\pi_1(GL_3^+(\mathbb{R})) = \pi_1(GL_n^+(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$ for any integer $n \geqslant 3$.

Proposition 2.1. *Let $\boldsymbol{\tau}_\ell \times \boldsymbol{\tau}_s$ and $\boldsymbol{\tau}_p$ be the frames of $T(S^1 \times D^2)$ and $\boldsymbol{\nu}_{S^1 \times D^2} : S^1 \times D^2 \to \nu(S^1 \times D^2, D^2 \times D^2)$ the section above.*

(1) *Any frame over $S^1 \times D^2$ of $T(S^1 \times D^2)$ is homotopic to the frame $\boldsymbol{\tau}_p$ or the frame $\boldsymbol{\tau}_\ell \times \boldsymbol{\tau}_s$.*

(2) *The frame $\boldsymbol{\nu}_{S^1 \times D^2} + (\boldsymbol{\tau}_\ell \times \boldsymbol{\tau}_s)$ over $S^1 \times D^2$ of $T(D^2 \times D^2) = T(D^2) \times T(D^2)$ does not extend to a frame over $D^2 \times D^2$ of $T(D^2 \times D^2)$.*

(3) *The frame $\boldsymbol{\nu}_{S^1 \times D^2} + \boldsymbol{\tau}_p$ is homotopic to the frame $(\boldsymbol{\tau}_s \times \boldsymbol{\tau}_s)|_{S^1 \times D^2}$. Hence $\boldsymbol{\nu}_{S^1 \times D^2} + \boldsymbol{\tau}_p$ extends to a frame over $D^2 \times D^2$ of $T(D^2 \times D^2)$.*

Each statement of (1)–(3) is true even if the term 'frame' is replaced by the term 'stable frame'.

From the fact that $\pi_1(GL_3^+(\mathbb{R})) = \pi_1(GL_n^+(\mathbb{R})) \cong \mathbb{Z}/2\mathbb{Z}$ and $\pi_2(GL_n^+(\mathbb{R})) = 0$ $(n \geqslant 3)$, one obtains the next proposition:

Proposition 2.2. *Let X be a CW-complex having the homotopy type of a finite, CW-complex of dimension $\leqslant 2$, and let η_i $(i = 1, 2)$ be oriented, real vector bundles over X such that $\eta_1 \supseteq \varepsilon_X(\mathbb{R}^3)$. Set $\eta = \eta_1 \oplus \eta_2$. If $\boldsymbol{\tau}$ and $\boldsymbol{\tau}_2$ are frames over X of η and η_2 respectively, then there exists a frame $\boldsymbol{\tau}_1$ over X of η_1 such that $\boldsymbol{\tau}_1 + \boldsymbol{\tau}_2$ is homotopic to $\boldsymbol{\tau}$. Moreover, such $\boldsymbol{\tau}_1$ is unique up to homotopy.*

Definition 2.3. Let $V = \coprod_{j=1}^n V_j$ be an oriented, 3-dimensional manifold with connected components V_j $(\dim V_j = 3)$ and $\boldsymbol{\kappa}$ a frame over V of $T(V)$ (i.e., $\boldsymbol{\kappa} : V \to P^+(T(V))$).

(1) For an orientation preserving, smooth immersion $\alpha : S^1 \times D^2 \to \mathrm{Int}(V)$, we define $\overline{\mu}_{\boldsymbol{\kappa}}(\alpha) \in \mathbb{Z}/2\mathbb{Z}$ by

$$\overline{\mu}_{\boldsymbol{\kappa}}(\alpha_i) = \begin{cases} 0 & \text{if } (\mathrm{d}\alpha_i)^*\boldsymbol{\kappa} \text{ is homotopic to } \boldsymbol{\tau}_p, \\ 1 & \text{if } (\mathrm{d}\alpha_i)^*\boldsymbol{\kappa} \text{ is homotopic to } \boldsymbol{\tau}_\ell \times \boldsymbol{\tau}_s. \end{cases}$$

(2) For an orientation preserving, smooth immersion

$$\alpha = \coprod_{i=1}^{m} \alpha_i \quad \left(\alpha_i : \left(S^1 \times D^2\right)_i \to \mathrm{Int}(V)\right)$$

we define $\overline{\mu}_{\boldsymbol{\kappa}}(\alpha) \in \mathbb{Z}/2\mathbb{Z}$ by

$$\overline{\mu}_{\boldsymbol{\kappa}}(\alpha) = \sum_{i=1}^{m} \overline{\mu}_{\boldsymbol{\kappa}}(\alpha_i).$$

Clearly, the value $\overline{\mu}_{\boldsymbol{\kappa}}(\alpha)$ is invariant under regular homotopies of α in $\mathrm{Int}(V)$. By Proposition 2.1, $\overline{\mu}_{\boldsymbol{\kappa}}(\alpha_i) = 0$ if and only if the frame $(\boldsymbol{\nu}_{S^1 \times D^2} + (\mathrm{d}\alpha_i)^*\boldsymbol{\kappa})|_{S^1 \times \{0\}}$ is extensible to a frame over D^2 of $T(D^2) \oplus \varepsilon_{D^2}(\mathbb{R}^2)$ $(= T(D^2 \times D^2)_{D^2 \times \{0\}})$.

3. Extensibility of frames

Let M be a compact, oriented, 2-dimensional manifold with boundary ∂M. We can take a manifold neighborhood $\mathrm{Col}(\partial M, M)$ of ∂M in M, such that

$$I \times \partial M \overset{\varphi}{\cong} \mathrm{Col}(\partial M, M),$$

$\varphi(\{1\} \times \partial M) = \partial M$ and $\varphi(1, x) = x$ for any $x \in \partial M$. Such $\mathrm{Col}(\partial M, M)$ is called a *collar neighborhood* of ∂M in M.

Now suppose $\partial M = S^1$ and fix an identification map $\mathrm{Col}(\partial M, M) = \mathrm{Col}(\partial D^2, D^2)$. Then the identification map $T(M)_{S^1} = T(D^2)_{S^1}$ $(S^1 = \partial M = \partial D^2)$ is induced.

Proposition 3.1. *Let M be a compact, connected, oriented, 2-dimensional manifold such that $\partial M = S^1$ and $\boldsymbol{\tau}$ a frame over S^1 of $T(M)_{S^1} \oplus \varepsilon_{S^1}(\mathbb{R}^k)$ $(k \geqslant 1)$. Then $\boldsymbol{\tau}$ is extensible to a frame over M of $T(M) \oplus \varepsilon_M(\mathbb{R}^k)$ if and only if $\boldsymbol{\tau}$ is extensible to a frame over D^2 of $T(D^2) \oplus \varepsilon_{D^2}(\mathbb{R}^k)$.*

The lemma below is used for our proof of the proposition above.

Let ξ and η be oriented, real vector bundles over locally compact, Hausdorff spaces X and Y respectively, with fiber $\mathbb{R}^n$. Let $f : X \to Y$ be a map. A map $b : \xi \to \eta$ is called a *vector bundle map* (or simply a *bundle map*) *covering* f if $b(\xi_x) \subset \eta_{f(x)}$ $(x \in X)$ and $b_x = b|_{\xi_x} : \xi_x \to \eta_{f(x)}$ is an orientation preserving, linear isomorphism. In such a case, we obtain the bundle isomorphism $f^{\%}b : \xi \to f^*\eta$ canonically induced from b and f. On the other hand, if $b : \xi \to f^*\eta$ is a bundle isomorphism then we obtain the bundle map $f_{\%}b : \xi \to \eta$ covering f canonically induced from b and f.

Lemma 3.2. *Let M be a compact, connected, oriented, 2-dimensional manifold such that $\partial M = S^1$, and let k be an integer $\geqslant 1$. Then there exists a degree 1, smooth* *map* $f:(M,\partial M)\to(D^2,\partial D^2)$ *and a vector bundle map* $b:T(M)\oplus f^*\varepsilon_{D^2}(\mathbb{R}^k)\to T(D^2)\oplus\varepsilon_{D^2}(\mathbb{R}^k)$ *covering* f, *such that*

$$f\big(\mathrm{Col}(\partial M,M)\big)=\mathrm{Col}\left(\partial D^2,D^2\right),$$
$$f\big(M\setminus\mathrm{Col}(\partial M,M)\big)=D^2\setminus\mathrm{Col}\left(\partial D^2,D^2\right),$$

$f|_{\mathrm{Col}(\partial M,M)}$ *coincides with the identification* $\mathrm{Col}(\partial M,M)=\mathrm{Col}(\partial D^2,D^2)$, *and*

$$b|_{\mathrm{Col}(\partial M,M)}=\mathrm{d}f|_{\mathrm{Col}(\partial M,M)}\oplus f^{\%}\,\mathrm{id}_{\varepsilon_{D^2}(\mathbb{R}^k)}\,.$$

The proof of the lemma will be given at the end of Section 4.

Proof of Proposition 3.1. First recall $\pi_1(GL^+_{2+k}(\mathbb{R}))\cong\mathbb{Z}/2\mathbb{Z}$. By Steenrod's obstruction theory (cf. [14,7]), the obstruction $d_M(\boldsymbol{\tau})$ to extending $\boldsymbol{\tau}$ to a frame over M of $T(M)\oplus\varepsilon_M(\mathbb{R}^k)$, lies in

$$H^2\big(M,\partial M;\pi_1\big(GL^+_{2+k}(\mathbb{R})\big)\big)\cong\mathbb{Z}/2\mathbb{Z}.$$

Replacing M by D^2, we obtain $d_{D^2}(\boldsymbol{\tau})$ in

$$H^2\big(D^2,\partial D^2;\pi_1\big(GL^+_{2+k}(\mathbb{R})\big)\big)\cong\mathbb{Z}/2\mathbb{Z}.$$

Let $f:M\to D^2$ and $b:T(M)\oplus\varepsilon_M(\mathbb{R}^k)\to T(D^2)\oplus\varepsilon_{D^2}(\mathbb{R}^k)$ be maps as in Lemma 3.2. Since f is degree 1, the induced homomorphism

$$f^*:H^2\big(D^2,\partial D^2;\pi_1\big(GL^+_{2+k}(\mathbb{R})\big)\big)\to H^2\big(M,\partial M;\pi_1\big(GL^+_{2+k}(\mathbb{R})\big)\big)$$

is an isomorphism. On the other hand, from the definition of obstructions, it follows that $f^*d_{D^2}(\boldsymbol{\tau})=d_M(\boldsymbol{\tau})$. Thus, $d_M(\boldsymbol{\tau})=0$ if and only if $d_{D^2}(\boldsymbol{\tau})=0$. This proves Proposition 3.1. □

Let M be a compact, oriented, 2-dimensional manifold. Define the section

$$\boldsymbol{\nu}_{\partial M\times D^2}:\partial M\times D^2\to T\big(M\times D^2\big)_{\partial M\times D^2}=\big(\nu(\partial M,M)\oplus T(\partial M)\big)\times T\big(D^2\big)$$

similarly to $\boldsymbol{\nu}_{S^1\times D^2}$ in Section 2. If $\boldsymbol{\tau}$ is a frame over $\partial M\times D^2$ of $T(M\times D^2)$, we canonically regard $\boldsymbol{\tau}|_{\partial M\times\{0\}}$ as a frame over ∂M of $T(M)\oplus\varepsilon_M(\mathbb{R}^2)$.

Proposition 3.3. *Let* $\boldsymbol{\tau}$ *(respectively* $\boldsymbol{\omega}$*) be a frame over* S^1 *(respectively* D^2*) of* $T(D^2)\oplus\varepsilon_{D^2}(\mathbb{R}^k)$ $(k\geqslant 1)$. *Let* $d_{S^1}(\boldsymbol{\tau},\boldsymbol{\omega}|_{S^1})$ *(respectively* $d_{D^2}(\boldsymbol{\tau})$*) denote the obstruction in* $H^1(S^1;G)$ *(respectively* $H^2(D^2,S^1;G)$*) to obtaining a homotopy* $\boldsymbol{\tau}\simeq\boldsymbol{\omega}|_{S^1}$ *(respectively a frame* $\boldsymbol{\tau}'$ *such that* $\boldsymbol{\tau}'|_{S^1}=\boldsymbol{\tau}$*) over* S^1 *(respectively* D^2*), where* $G=\pi_1(GL^+_{2+k}(\mathbb{R}))$. *Then* $d_{S^1}(\boldsymbol{\tau},\boldsymbol{\omega}|_{S^1})$ *is determined independently of the choice of a frame* $\boldsymbol{\omega}$ *over* D^2. *If* $\delta:H^1(S^1;G)\to H^2(D^2,S^1;G)$ *denotes the connecting homomorphism of Mayer–Vietoris exact sequence, then* $\delta(d_{S^1}(\boldsymbol{\tau},\boldsymbol{\omega}|_{S^1}))=d_{D^2}(\boldsymbol{\tau})$.

Proof. The first conclusion of the proposition follows from the fact that any two frames over D^2 are homotopic to each other.

Next note that $H^2(D^2, S^1; G) \cong \mathbb{Z}/2\mathbb{Z}$ and $H^1(S^1; G) \cong \mathbb{Z}/2\mathbb{Z}$. Since $H^1(D^2; G) = 0$ and $H^2(S^1; G) = 0$, δ is the unique isomorphism. The frame $\boldsymbol{\tau}$ is extensible to a frame over D^2 if and only if $\boldsymbol{\tau}$ is homotopic to $\boldsymbol{\omega}|_{S^1}$. Thus one gets

$$\delta\big(d_{S^1}(\boldsymbol{\tau}, \boldsymbol{\omega}|_{S^1})\big) = d_{D^2}(\boldsymbol{\tau}). \qquad \square$$

Theorem 3.4. *Let $V = \coprod_{j=1}^{n} V_j$ be an oriented, 3-dimensional manifold with connected components V_j, $\boldsymbol{\kappa}$ a frame over V of $T(V)$, $\alpha : K \to \mathrm{Int}(V)$ ($K = \coprod_{i=1}^{m} K_i$, $K_i = (S^1 \times D^2)_i$) an orientation preserving, smooth immersion, and M a compact, connected, oriented, 2-dimensional manifold such that*

$$\partial M = \coprod_{i=1}^{m} S_i^1 \quad (S_i^1 = S^1 \times \{0\} \subset K_i).$$

Set $\boldsymbol{\tau} = \boldsymbol{\nu}_{\partial M \times D^2} + (\mathrm{d}\alpha)^* \boldsymbol{\kappa}$ and identify ∂M with $\partial M \times \{0\}$ in the canonical way. Then $\boldsymbol{\tau}|_{\partial M}$ is extensible to a frame over M of $T(M) \oplus \varepsilon_M(\mathbb{R}^2)$ if and only if $\overline{\mu}_{\boldsymbol{\kappa}}(\alpha) = 0$.

Corollary 3.5. *Let V, $\boldsymbol{\kappa}$, α, M, and $\boldsymbol{\tau}$ be as in Theorem 3.4. Let M' be a compact, connected, oriented, 2-dimensional, smooth manifold such that $\partial M' = \coprod_{i=1}^{m} S_i^1 = \partial M$. Then $\boldsymbol{\tau}|_{\partial M}$ is extensible to a frame over M of $T(M) \oplus \varepsilon_M(\mathbb{R}^2)$ if and only if $\boldsymbol{\tau}|_{\partial M'}$ is extensible to a frame over M' of $T(M') \oplus \varepsilon_{M'}(\mathbb{R}^2)$.*

Proof of Theorem 3.4. Take a 2-dimensional disk $D = D^2$ embedded in the interior of M and set $L = M \setminus \mathrm{Int}(D)$ and $G = \pi_1(GL_4^+(\mathbb{R}))$ ($\cong \mathbb{Z}/2\mathbb{Z}$). Since $H^2(M; G) = 0$ (respectively $H^2(L, \partial L; G) = 0$), there exists a frame $\boldsymbol{\omega}$ (respectively $\boldsymbol{\tau}'$ such that $\boldsymbol{\tau}'|_{\partial M} = \boldsymbol{\tau}|_{\partial M}$) over M (respectively L) of $T(M) \oplus \varepsilon_M(\mathbb{R}^2)$. The obstruction $d_{\partial M}(\boldsymbol{\tau}|_{\partial M}, \boldsymbol{\omega}|_{\partial M})$ to obtaining a homotopy $\boldsymbol{\tau}|_{\partial M} \simeq \boldsymbol{\omega}|_{\partial M}$ lies in

$$H^1(\partial M; G) = \bigoplus_{i=1}^{m} H^1(S_i^1; G) \cong \bigoplus_{i=1}^{m} G_i$$

(where $G_i = \mathbb{Z}/2\mathbb{Z}$), $d_{\partial D}(\boldsymbol{\tau}'|_{\partial D}, \boldsymbol{\omega}|_{\partial D})$ lies in $H^1(\partial D; G) \cong \mathbb{Z}/2\mathbb{Z}$, and $d_L(\boldsymbol{\tau}', \boldsymbol{\omega}|_L)$ lies in $H^1(L; G) \cong \mathbb{Z}/2\mathbb{Z}$ (note that $H^2(L; \pi_2(GL_4^+(\mathbb{R}))) = 0$). The obstructions $d_{\partial M}(\boldsymbol{\tau}|_{\partial M}, \boldsymbol{\omega}|_{\partial M})$ and $d_{\partial D}(\boldsymbol{\tau}'|_{\partial D}, \boldsymbol{\omega}|_{\partial D})$ are obtained from $d_L(\boldsymbol{\tau}', \boldsymbol{\omega}|_L)$ via restriction respectively. Clearly, $H^2(M, \partial M; G) \cong \mathbb{Z}/2\mathbb{Z}$, $H^2(M, L; G) = H^2(D, \partial D) \cong \mathbb{Z}/2\mathbb{Z}$ and $H^1(\partial D; G) \cong \mathbb{Z}/2\mathbb{Z}$. Let $\delta_{(D,\partial D)} : H^1(\partial D; G) \to H^2(D, \partial D; G)$ denote the connecting homomorphism of the Mayer–Vietoris exact sequence. Then $\delta_{(D,\partial D)}$ is the unique isomorphism. By Proposition 3.3,

$$\delta_{(D,\partial D)}\big(d_{\partial D}(\boldsymbol{\tau}'|_{\partial D}, \boldsymbol{\omega}|_{\partial D})\big) = d_D(\boldsymbol{\tau}'|_{\partial D}).$$

Let

$$\delta_{(M,\partial M)} : H^1(\partial M) = \bigoplus_{i=1}^{m} \mathbb{Z}/2\mathbb{Z} \to H^2(M, \partial M) = \mathbb{Z}/2\mathbb{Z}$$

denote the connecting homomorphism of the Mayer–Vietoris exact sequence. Then

$$\partial_{(M,\partial M)}(a_1, \ldots, a_m) = a_1 + \cdots + a_m$$

for $a_i \in \mathbb{Z}/2\mathbb{Z}$. With the identification $H^1(\partial M) = \bigoplus_{i=1}^m \mathbb{Z}/2\mathbb{Z}$, we have

$$d_{\partial M}(\boldsymbol{\tau}|_{\partial M}, \boldsymbol{\omega}|_{\partial M}) = \left(d_{S_i^1}(\boldsymbol{\tau}|_{S_i^1}, \boldsymbol{\omega}|_{S_i^1})\right)_{i=1}^m \in \bigoplus_{i=1}^m \mathbb{Z}/2\mathbb{Z}.$$

Now chasing the commutative diagram

$$\begin{tikzcd} H^1(\partial M) \arrow[d, "\delta_{(M,\partial M)}"'] & H^1(L) \arrow[l] \arrow[r] \arrow[d, "\delta_{(M,L)}"'] & H^1(\partial D) \arrow[d, "\delta_{(D,\partial D)}"'] \\ H^2(M,\partial M) & H^2(M,L) \arrow[l, "\cong"'] \arrow[r, "\cong"] & H^2(D,\partial D) \end{tikzcd},$$

one obtains

$$\begin{aligned} d_M(\boldsymbol{\tau}|_{\partial M}) &= \delta_{(M,\partial M)}\left(d_{\partial M}(\boldsymbol{\tau}|_{\partial M}, \boldsymbol{\omega}|_{\partial M})\right) \\ &= \sum_{i=1}^m \delta_{(M,\partial M)}\left(d_{S_i^1}(\boldsymbol{\tau}|_{S_i^1}, \boldsymbol{\omega}|_{S_i^1})\right) \in \mathbb{Z}/2\mathbb{Z}. \end{aligned}$$

By Proposition 2.1(3),

$$d_{S_i^1}(\boldsymbol{\tau}|_{S_i^1}, \boldsymbol{\omega}|_{S_i^1}) = d_{S_i^1}\left(\boldsymbol{\tau}|_{S_i^1}, (\boldsymbol{\nu}_{S^1\times D^2} + \boldsymbol{\tau}_p)|_{S^1\times\{0\}}\right)$$

with the canonical identification $S_i^1 = S^1 \times \{0\}$. Since $\boldsymbol{\tau} = \boldsymbol{\nu}_{\partial M\times D^2} + (\mathrm{d}\alpha)^*\boldsymbol{\kappa}$, using Definition 2.3 one can check

$$d_{S_i^1}\left(\boldsymbol{\tau}|_{S_i^1}, (\boldsymbol{\nu}_{S^1\times D^2} + \boldsymbol{\tau}_p)|_{S^1\times\{0\}}\right) = \overline{\mu}_{\boldsymbol{\kappa}}(\alpha_i) \in \mathbb{Z}/2\mathbb{Z}.$$

Putting all together, one obtains

$$d_M(\boldsymbol{\tau}|_{\partial M}) = \delta_{(M,\partial M)}\left(d_{\partial M}(\boldsymbol{\tau}|_{\partial M}, \boldsymbol{\omega}|_{\partial M})\right) = \sum_{i=1}^m \overline{\mu}_{\boldsymbol{\kappa}}(\alpha_i) = \overline{\mu}_{\boldsymbol{\kappa}}(\alpha) \in \mathbb{Z}/2\mathbb{Z}.$$

This proves Theorem 3.4. □

4. Surgery

Let us begin with the definition of the involution σ_k on $\mathbb{R}^k$ ($k \geqslant 1$):

$$\sigma_k(x_1, \ldots, x_{k-1}, x_k) = (x_1, \ldots, x_{k-1}, -x_k).$$

Usually, we write $\overline{x}$ for $\sigma_k(x)$ ($x \in \mathbb{R}^k$). The complex number field $\mathbb{C}$ is identified with $\mathbb{R}^2$ via the map $x + \sqrt{-1}y \mapsto (x, y)$ (x, $y \in \mathbb{R}$). Thus the involution σ_2 coincides with the complex conjugation.

If a subset A of $\mathbb{R}^k$ is closed under σ_k and α is a map $A \to X$, then define $\alpha^\&$ by $\alpha^\& = \{1\} \times (\alpha \cdot \sigma_k|_A) : A \to \{1\} \times X$. Let X be an oriented, n-dimensional manifold and $\alpha : S^k \times D^{n-k} \to \mathrm{Int}(X)$ (the interior of X) an orientation preserving, smooth embedding. The attaching space

$$\mathrm{Trace}(X, \alpha) := (I \times X) \bigcup_{\alpha^\&} (D^{k+1} \times D^{n-k})$$

is called the *trace of surgery along* α. Smoothing corners, we regard the trace of surgery as a smooth manifold. Further the trace of surgery has the orientation whose restriction to $I \times X$ coincides with $[I \times X]$. Decompose $\partial\,\mathrm{Trace}(X, \alpha)$ into three submanifolds $\{0\} \times X$, $I \times \partial X$ and $\mathrm{Surg}(X, \alpha)$ such that

$$\big((\{0\} \times X) \cup (I \times \partial X)\big) \cap \mathrm{Surg}(X, \alpha) = \partial\,\mathrm{Surg}(X, \alpha) = \{1\} \times \partial X.$$

The oriented manifold $\mathrm{Surg}(X, \alpha)$ is called the *manifold resulting from surgery along* α. This procedure to obtain $\mathrm{Surg}(X, \alpha)$ is called k*-dimensional surgery on* X *along* α. We define the map

$$\mathrm{dual}\,\alpha : S^{n-k} \times D^{k+1} \to D^{k+1} \times D^{n-k} \subset \mathrm{Surg}(X, \alpha)$$

called the *dual to* α by

$$\mathrm{dual}\,\alpha(x, y) = (\overline{y}, \overline{x}) \quad \text{for } x \in S^{n-k} \text{ and } y \in D^{k+1}.$$

Now consider the case where $n = 3$ and $k = 1$. The map $\alpha|_{S^1 \times \{1\}} : S^1 \to \mathrm{Im}(\alpha)$; $x \mapsto \alpha(x, 1)$ (respectively $\alpha|_{\{1\} \times S^1} : S^1 \to \mathrm{Im}(\alpha)$; $x \mapsto \alpha(1, x)$) is called the *longitude* (respectively *meridian*) of α. In the manifold $I \times X$ ($\subset \mathrm{Trace}(X, \alpha)$), we have

$$\mathrm{dual}\,\alpha(x, 1) = \alpha^{\&}(1, \overline{x}) = \big(1, \alpha(1, x)\big) \quad \text{and}$$
$$\mathrm{dual}\,\alpha(1, \overline{x}) = \alpha^{\&}(x, 1) = \big(1, \alpha(x, 1)\big)$$

for all $x \in S^1$, where $\overline{x}$ is the complex conjugate to x. Thus, we obtain the next proposition:

Proposition 4.1. *Let* $\alpha : S^1 \times D^2 \to \mathrm{Int}(X)$ *be an orientation preserving, smooth embedding. Then the meridian of* α *is the longitude of* $\mathrm{dual}\,\alpha$ *and the longitude of* α *is the reversed meridian of* $\mathrm{dual}\,\alpha$.

Coming back to a general case, let $\alpha : S^k \times D^{n-k} \to \mathrm{Int}(X)$ be an orientation preserving, smooth embedding, Y an n-dimensional manifold and $f : X \to Y$ a map. If for a map $\beta : D^{k+1} \times D^{n-k} \to Y$, the diagram

$$\begin{array}{ccc} S^k \times D^{n-k} & \xrightarrow{\alpha} & X \\ {\scriptstyle \text{canonical}} \downarrow & & \downarrow {\scriptstyle f} \\ D^{k+1} \times D^{n-k} & \xrightarrow{\beta} & Y \end{array} \qquad (\mathcal{M})$$

commutes, then define the $\mathrm{trace}(f, \mathcal{M}) : \mathrm{Trace}(X, \alpha) \to I \times Y$ by

$$\begin{cases} \mathrm{trace}(f, \mathcal{M})|_{I \times X} = \mathrm{id}_I \times f, \\ \mathrm{trace}(f, \mathcal{M})|_{D^{k+1} \times D^{n-k}} = \beta^{\&}. \end{cases}$$

Define the map $\mathrm{surg}(f, \mathcal{M}) : \mathrm{Surg}(X, \alpha) \to Y$ by

$$\mathrm{trace}(f, \mathcal{M})|_{\mathrm{Surg}(X, \alpha)} = \{1\} \times \mathrm{surg}(f, \mathcal{M}).$$

We call the map $\mathrm{surg}(f, \mathcal{M})$ the *map resulting from surgery of* f *along* α (or *with data* $\mathcal{M}$).

Next we discuss bundle data. For a while we argue in the category of smooth manifolds possibly with corners. Define $\nu(S^k \times D^{n-k}, D^{k+1} \times D^{n-k})$ to be the subbundle of $T(D^{k+1} \times D^{n-k})_{S^k \times D^{n-k}}$ such that

$$\begin{aligned}\nu\left(S^k \times D^{n-k}, D^{k+1} \times D^{n-k}\right) &= \nu\left(S^k, D^{k+1}\right) \times D^{n-k} \\ &\subseteq T\left(D^{k+1}\right)_{S^k} \times T\left(D^{n-k}\right) \\ &= T\left(D^{k+1} \times D^{n-k}\right)_{S^k \times D^{n-k}}.\end{aligned}$$

Using the canonical identification $\nu(S^k, D^{k+1}) = \varepsilon_{S^k}(\mathbb{R})$, we obtain the identification

$$\nu\left(S^k \times D^{n-k}, D^{k+1} \times D^{n-k}\right) = \varepsilon_{S^k \times D^{n-k}}(\mathbb{R}).$$

For a smooth embedding $\alpha : S^k \times D^{n-k} \to \operatorname{Int}(X)$, define

$$(\mathrm{d}\alpha)^\star : T\left(D^{k+1} \times D^{n-k}\right)_{S^k \times D^{n-k}} \to T(I \times X)_{\{1\} \times X}$$

to be the map making the diagram

$$\begin{tikzcd}
T\left(D^{k+1} \times D^{n-k}\right)_{S^k \times D^{n-k}} \arrow[r, "(\mathrm{d}\alpha)^\star"] \arrow[d, "\text{canonical}"'] & T(I \times X)|_{\{1\} \times X} \\
\nu\left(S^k \times D^{n-k}, D^{k+1} \times D^{n-k}\right) \oplus T\left(S^k \times D^{n-k}\right) \arrow[r, "-(\alpha^\&)^\% \mathrm{id} \oplus \mathrm{d}(\alpha^\&)"] & \{1\} \times \left(\varepsilon_X(\mathbb{R}) \oplus T(X)\right) \arrow[u, "\text{canonical}"]
\end{tikzcd}$$

commute. Then it follows that

$$T\left(\operatorname{Trace}(X, \alpha)\right) = T(I \times X) \bigcup_{(\mathrm{d}\alpha)^\star} T\left(D^{k+1} \times D^{n-k}\right).$$

Let η_+ and η_- be oriented, real vector bundles over Y and let $b : T(X) \oplus f^*\eta_- \to \eta_+$ be a bundle map covering the map f above. We suppose that

$$\eta_+ \supseteq \varepsilon_Y\left(\mathbb{R}^{n+1}\right). \tag{4.2}$$

For a bundle map $\gamma : T(D^{k+1} \times D^{n-k}) \oplus \beta^*\eta_- \to \varepsilon_Y(\mathbb{R}) \oplus \eta_+$ covering β, define the bundle map

$$\gamma^\star : T\left(D^{k+1} \times D^{n-k}\right) \oplus \beta^*\eta_- \to \{1\} \times \left(\varepsilon_Y(\mathbb{R}) \oplus \eta_+\right)$$

by

$$\begin{aligned}&\gamma^\star(u, v) = \left(1, -\gamma_1\left(\mathrm{d}\sigma_{k+1}(u), v\right), \gamma_2\left(\mathrm{d}\sigma_{k+1}(u), v\right)\right) \\ &\quad \left((u, v) \in T\left(D^{k+1} \times D^{n-k}\right) \oplus \beta^*\eta_-,\right. \\ &\quad\quad \left.\gamma(u, v) = \left(\gamma_1(u, v), \gamma_2(u, v)\right) \in \varepsilon_Y(\mathbb{R}) \oplus \eta_+\right).\end{aligned}$$

If the diagram

$$\begin{tikzcd}
\nu \oplus T\left(S^k \times D^{n-k}\right) \oplus (f\alpha)^*\eta_- \arrow[r, "\tilde{\alpha}"] \arrow[d, "\text{canonical}"'] & \varepsilon_X(\mathbb{R}) \oplus T(X) \oplus f^*\eta_- \arrow[d, "\text{canonical} \oplus b"'] \\
T\left(D^{k+1} \times D^{n-k}\right) \oplus \beta^*\eta_- \arrow[r, "\gamma"] & \varepsilon_Y(\mathbb{R}) \oplus \eta_+
\end{tikzcd} \tag{$\mathcal{B}$}$$

$$\big(\nu := \nu\big(S^k \times D^{n-k}, D^{k+1} \times D^{n-k}\big) = \varepsilon_{S^k \times D^{n-k}}(\mathbb{R}),$$
$$\widetilde{\alpha} = \text{canonical} \oplus d\alpha \oplus \text{canonical}\big)$$

commutes, then define

$$\operatorname{trace}(b, \mathcal{B}) : T\big(\operatorname{Trace}(X, \alpha)\big) \oplus F^*(I \times \eta_-) \to \varepsilon_I(\mathbb{R}) \times \eta_+$$
$$\big(= \varepsilon_{I \times Y}(\mathbb{R}) \oplus (I \times \eta_+)\big)$$

$(F = \operatorname{trace}(f, \mathcal{M}))$ by

$$\begin{cases} \operatorname{trace}(b, \mathcal{B})_{I \times X} = \mathrm{id}_{\varepsilon_I(\mathbb{R})} \times b, \\ \operatorname{trace}(b, \mathcal{B})|_{D^{k+1} \times D^{n-k}} = \gamma^*. \end{cases}$$

After smoothing corners of $\operatorname{Trace}(X, \alpha)$, we have obtained the smooth manifold $\operatorname{Surg}(X, \alpha)$. Thus, applying the procedure of smoothing corners to $\operatorname{trace}(b, \mathcal{B})$ and restricting the result to $\operatorname{Surg}(X, \alpha)$, we obtain a bundle map

$$b'' : \varepsilon_{\operatorname{Surg}(X,\alpha)}(\mathbb{R}) \oplus T\big(\operatorname{Surg}(X, \alpha)\big) \oplus \operatorname{surg}(f, \mathcal{M})^* \eta_- \to \{1\} \times \big(\varepsilon_Y(\mathbb{R}) \oplus \eta_+\big)$$

covering $\operatorname{surg}(f, \mathcal{M})$. By [3, Proposition 10.1], it follows from hypothesis (4.2) that b'' is regularly homotopic to a bundle map

$$\operatorname{surg}(f, \mathcal{M})^{\%} \mathrm{id}_{\varepsilon_Y(\mathbb{R})} \oplus \big(\{1\} \times b'\big)$$

$(b' : T(\operatorname{Surg}(X, \alpha)) \oplus \operatorname{surg}(f, \mathcal{M})^* \eta_- \to \eta_+)$ covering $\operatorname{surg}(f, \mathcal{M})$.

Remark 4.3. If $\operatorname{Surg}(X, \alpha)$ is homotopy equivalent to an ℓ-dimensional CW-complex, then by [3, Proposition 10.1] it is sufficient for the existence of b' above that $\eta_+ \supseteq \varepsilon_Y(\mathbb{R}^{\ell+1})$.

Let

$$\operatorname{surg}(b, \mathcal{B}) : T\big(\operatorname{Surg}(X, \alpha)\big) \oplus \operatorname{surg}(f, \mathcal{M})^* \eta_- \to \eta_+$$

denote a bundle map b' above.

If $\alpha_1, \ldots, \alpha_m : S^k \times D^{n-k} \to \operatorname{Int}(X)$ are orientation preserving, smooth embeddings which are mutually disjoint, then performing surgery along α_i simultaneously, we obtain the trace of surgery

$$\operatorname{Trace}(X, \alpha) := (I \times X) \bigcup_{\alpha^\&} \coprod_{i=1}^{m} \big(D^{k+1} \times D^{n-k}\big)_i,$$

and the resulting manifold $\operatorname{Surg}(X, \alpha)$ $(\alpha = \alpha_1 \cup \cdots \cup \alpha_m,\ \alpha^\& = \alpha_1^\& \cup \cdots \cup \alpha_m^\&)$. If furthermore for each $i = 1, \ldots, m$, $(\mathcal{M}_i)$ is a commutative diagram for α_i and β_i similar to $(\mathcal{M})$ for α and β there, then we obtain the map $\operatorname{trace}(f, \mathcal{M}) : \operatorname{Trace}(X, \alpha) \to I \times Y$ and the map $\operatorname{surg}(f, \mathcal{M}) : \operatorname{Surg}(X, \gamma) \to Y$ $(\mathcal{M} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_m$ and $\alpha = \alpha_1 \cup \cdots \cup \alpha_m)$. Similarly, we construct $\operatorname{surg}(b, \mathcal{B}) : T(\operatorname{Surg}(X, \alpha)) \oplus \operatorname{surg}(f, \mathcal{M})^* \eta_- \to \eta_+$ covering $\operatorname{surg}(f, \mathcal{M})$ for $\alpha = \alpha_1 \cup \cdots \cup \alpha_m$, $\mathcal{M} = \mathcal{M}_1 \cup \cdots \cup \mathcal{M}_m$ and $\mathcal{B} = \mathcal{B}_1 \cup \cdots \cup \mathcal{B}_m$.

Proof of Lemma 3.2. First note that M is a punctured surface and hence is obtained from D^2 by 0-dimensional surgery. To obtain a degree 1, smooth map $f : M \to D^2$ and a bundle map $b : T(M) \oplus f^*\varepsilon_{D^2}(\mathbb{R}^k) \to T(D^2) \oplus \varepsilon_{D^2}(\mathbb{R}^k)$ covering f, we start with the surgery object

$$\left(D^2, \mathrm{id}_{D^2} : D^2 \to D^2, \mathrm{id}_{T(D^2)\oplus\varepsilon_{D^2}(\mathbb{R}^k)} : T(D^2) \oplus \varepsilon_{D^2}(\mathbb{R}^k) \to T(D^2) \oplus \varepsilon_{D^2}(\mathbb{R}^k)\right).$$

It suffices to show that the 0-dimensional surgery above can be performed together with the data id_{D^2} and $\mathrm{id}_{T(D^2)\oplus\varepsilon_{D^2}(\mathbb{R}^k)}$. Given an orientation preserving, smooth embedding $\alpha : S^0 \times D^2 \to \mathrm{Int}(D^2)$, we can choose a map $\beta : D^1 \times D^2 \to \mathrm{Int}(D^2)$ such that diagram ($\mathcal{M}$) commutes for $f = \mathrm{id}_{D^2}$, because D^2 is connected. Since D^2 and $GL^+_{2+k}(\mathbb{R})$ ($k \geqslant 1$) are connected, we can find

$$\gamma : T(D^1 \times D^2) \oplus \beta^*\varepsilon_{D^2}(\mathbb{R}^k) \to \varepsilon_{D^2}(\mathbb{R}) \oplus T(D^2) \oplus \varepsilon_{D^2}(\mathbb{R}^k)$$

such that diagram ($\mathcal{B}$) commutes for $\eta_+ = T(D^2) \oplus \varepsilon_{D^2}(\mathbb{R}^k)$, $\eta_- = \varepsilon_{D^2}(\mathbb{R}^k)$, and $b = \mathrm{id}_{T(D^2)\oplus\varepsilon_{D^2}(\mathbb{R}^k)}$. Moreover, a manifold obtained from D^2 by 0-dimensional surgery is homotopy equivalent to a 2-dimensional CW-complex. By Remark 4.3, the 0-dimensional surgery above is possible even if the data id_{D^2} and $\mathrm{id}_{T(D^2)\oplus\varepsilon_{D^2}(\mathbb{R}^k)}$ are attached. $\square$

5. Homology invariance of $\overline{\mu}$

Let V be a compact, oriented, 3-dimensional manifold and let

$$\alpha^{(i)} = \coprod_{j\in J(i)} \alpha^{(i)}_j \quad \left(i = 1, 2, \alpha^{(i)}_j : S^1_{i,j} \times D^2 \to \mathrm{Int}(V),\ S^1_{i,j} = S^1\right)$$

be disjoint, orientation preserving, smooth embeddings. These $\alpha^{(1)}$ and $\alpha^{(2)}$ are said to be *V-cobordant* to each other if there exist a compact, oriented, 2-dimensional, manifold B and an immersion $\beta = \beta_B : B \to \mathrm{Int}(V)$, and the following properties are satisfied:

$$\partial B = \partial_1 B \sqcup \partial_2 B, \qquad \partial_1 B = \coprod_{j\in J(1)} \partial_1 B_j, \qquad \partial_2 B = \coprod_{j\in J(2)} \partial_2 B_j,$$

$$\partial_1 B_j = \sigma_2\left(S^1_{1,j}\right) \quad \left(j \in J(1)\right), \qquad \partial_2 B_j = S^1_{2,j} \quad \left(j \in J(2)\right),$$

$$\beta\left(\mathrm{Col}(\partial_i B, B)\right) \subset \mathrm{Col}\left(\partial\, \mathrm{Im}\left(\alpha^{(i)}\right), V'\right) \quad (i = 1, 2),$$

$$\beta|_{\mathrm{Col}(\partial_1 B_j, B)} = \mathrm{id}_I \times \left(\mathrm{longitude}\left(\alpha^{(1)'}_j\right)\right) \quad \left(j \in J(1)\right),$$

$$\beta|_{\mathrm{Col}(\partial_2 B_j, B)} = \mathrm{id}_I \times \mathrm{longitude}\left(\alpha^{(2)}_j\right) \quad \left(j \in J(2)\right),$$

where

$$V' = \mathrm{Int}(V) \setminus \mathrm{Int}\left(\mathrm{Im}\left(\alpha^{(1)} \cup \alpha^{(2)}\right)\right),$$

we identify the collar neighborhood $\mathrm{Col}(\partial_i B_j, B)$ with $I \times \partial_i B_j$, $\alpha_j^{(1)'} = \alpha_j^{(1)} \cdot (\sigma_2 \times \sigma_2)$ and this $\sigma_2 \times \sigma_2$ is the map $S^1_{1,j} \times D^2 \to S^1_{1,j} \times D^2$.

Proposition 5.1. *Let V be a compact, connected, oriented, 3-dimensional, manifold with a frame $\boldsymbol{\kappa}: V \to P^+(T(V))$, and let*

$$\alpha^{(i)} = \coprod_{j \in J(i)} \alpha_j^{(i)} \quad \Bigl(\alpha_j^{(i)}: \coprod_{j \in J(i)} (S^1_{i,j} \times D^2) \to \mathrm{Int}(V)\Bigr)$$

be disjoint, orientation preserving, smooth embeddings ($i = 1, 2$). If $\alpha^{(1)}$ and $\alpha^{(2)}$ are V-cobordant to each other then

$$\overline{\mu}_{\boldsymbol{\kappa}}(\alpha^{(1)}) = \overline{\mu}_{\boldsymbol{\kappa}}(\alpha^{(2)}).$$

In particular, if $\alpha^{(2)}$ is null V-cobordant then $\overline{\mu}_{\boldsymbol{\kappa}}(\alpha^{(2)}) = 0$.

Proof. Set $\alpha_j^{(1)'} = \alpha_j^{(1)} \cdot (\sigma_2 \times \sigma_2)$ as above. Then $\mathrm{d}\alpha_j^{(1)'*} \boldsymbol{\kappa}$ is homotopic to $\mathrm{d}\alpha_j^{(1)*} \boldsymbol{\kappa}$. Thus, replacing $(J(1), J(2))$ by $(\emptyset, J(1) \cup J(2))$, it suffices to prove $\overline{\mu}_{\boldsymbol{\kappa}}(\alpha^{(2)}) = 0$ under the condition $J(1) = \emptyset$. Let B and β be the data of a null cobordism of $\alpha^{(2)}$ as above. We use α, S^1_j, ∂B and J in place of $\alpha^{(2)}$, $S^1_{2,j}$, $\partial_2 B$ and $J(2)$. Let $k_{j,\varepsilon}: S^1_j \to S^1_j \times \{\varepsilon\}$ ($\varepsilon = 0, 1$) denote the canonical identifications. For clarity of notation we write $D^2 \subset \mathbb{R}^2 = \mathbb{R}_1 \oplus \mathbb{R}_2$. Then the frame $k_{j,0}{}^*((\mathrm{d}\alpha^* \boldsymbol{\kappa})|_{S^1_j \times \{0\}})$ of $k_{j,0}{}^*(T(S^1_j) \times T(D^2)) = T(S^1_j) \oplus \varepsilon_{S^1_j}(\mathbb{R}_1 \oplus \mathbb{R}_2)$ is homotopic to the frame $k_{j,1}{}^*((\mathrm{d}\alpha^* \boldsymbol{\kappa})|_{S^1_j \times \{1\}})$. Let $b(\beta): T(B) \oplus \nu(\beta, V) \to T(V)$ denote the canonical bundle map covering β. Then $b(\beta)$ induces the frame $b(\beta)^* \boldsymbol{\kappa}$ over B of $T(B) \oplus \nu(\beta, V)$. There are the canonical identifications $\nu(\partial B, B) = \varepsilon_{\partial B}(\mathbb{R}_1)$ and $\nu(\beta, V)_{\partial B} = \varepsilon_{\partial B}(\mathbb{R}_2)$. The frame $(b(\beta)^* \boldsymbol{\kappa})|_{S^1_j}$ coincides with $k_{j,1}{}^*((\mathrm{d}\alpha_j{}^* \boldsymbol{\kappa})|_{S^1_j \times \{1\}})$. By Theorem 3.4, $\overline{\mu}_{\boldsymbol{\kappa}}(\alpha) = 0$ if and only if the frame

$$\coprod_{j \in J} \bigl((\boldsymbol{\nu}_{S^1_j \times D^2})|_{S^1_j \times \{0\}} + (b(\beta)^* \boldsymbol{\kappa})|_{S^1_j}\bigr)$$

is extensible to a frame over a compact, connected, oriented, 2-dimensional manifold M of $T(M) \oplus \varepsilon_M(\mathbb{R}_1 \oplus \mathbb{R}_2)$ such that $\partial M = \coprod_{j \in J} S^1_j$. We shall observe this for $M = B$. But to do so, the region $\nu(\partial M, M)$ of the normal section

$$\boldsymbol{\nu}_{\partial M} = (\boldsymbol{\nu}_{\partial M \times D^2})|_{\partial M \times \{0\}} = \coprod_{j \in J} (\boldsymbol{\nu}_{S^1_j \times D^2})|_{S^1_j \times \{0\}}$$

must be distinguished from the bundle previously identified with $\varepsilon_{\partial B}(\mathbb{R}_1)$. Thus we write $\varepsilon_{\partial M}(\mathbb{R}_0)$ for $\nu(\partial M, M)$. Now set $M = B$. Then $(b(\beta)^* \boldsymbol{\kappa})|_{\partial B}$ obviously extends to the frame $b(\beta)^* \boldsymbol{\kappa}$ over M of $b(\beta)^* T(V)$. It is also clear that the frame $\coprod_{j \in J} (\boldsymbol{\nu}_{S^1_j \times D^2})|_{S^1_j \times \{0\}}$ over ∂B of $\varepsilon_{\partial M}(\mathbb{R}_0)$ is extensible to a frame over M of $\varepsilon_M(\mathbb{R}_0)$. Thus we obtain $\overline{\mu}_{\boldsymbol{\kappa}}(\alpha) = 0$. $\square$

In the remainder of the paper, the coefficient ring of homology and cohomology groups are the ring of integers $\mathbb{Z}$, unless specifically mentioned. Let

$$\alpha = \coprod_{j \in J} \alpha_j \quad (\alpha_j: S^1 \times D^2 \to \mathrm{Int}(V))$$

be a smooth embedding. Then the sum of longitudes of α_j ($j \in J$) represents a homology element in $H_1(V \setminus \mathrm{Int}(\mathrm{Im}(\alpha)))$ and let $[\alpha]$ denote the element. Let j_α denote the natural homomorphism $H_1(\partial V) \to H_1(V \setminus \mathrm{Int}(\mathrm{Im}(\alpha)))$.

Theorem 5.2. *Let V and $\boldsymbol{\kappa}$ be as in Proposition 5.1 and let*

$$\alpha^{(i)} = \coprod_{j \in J(i)} \alpha_j^{(i)} \quad \left(i = 1, 2, \alpha_j^{(i)} : S^1_{i,j} \times D^2 \to \mathrm{Int}(V),\ S^1_{i,j} = S^1\right)$$

be disjoint smooth embeddings. If there exists an element $x \in H_1(\partial V)$ such that for each $i = 1$ and 2, $j_{\alpha^{(i)}*}(x) = [\alpha^{(i)}]$ in $H_1(V \setminus \mathrm{Int}(\mathrm{Im}(\alpha^{(i)})))$, then

$$\overline{\mu}_{\boldsymbol{\kappa}}\left(\alpha^{(1)}\right) = \overline{\mu}_{\boldsymbol{\kappa}}\left(\alpha^{(2)}\right).$$

Proving the theorem, we use the following two lemmas which are basic in algebraic and differential topology.

Let M be a compact, connected, oriented, n-dimensional, manifold possibly with boundary $\partial M = \partial_1 M \coprod \partial_2 M$. The set of all homotopy classes of maps $(M, \partial_1 M) \to (S^1, \{-1\})$ is denoted by $[(M, \partial_1 M), (S^1, \{-1\})]$. There is a canonical map

$$\rho_{M, \partial_1 M} : \left[(M, \partial_1 M), \left(S^1, \{-1\}\right)\right] \to H^1(M, \partial_1 M)$$

defined by $\rho_{M,\partial_1 M}([h]) = h^* a$ ($h : (M, \partial_1 M) \to (S^1, \{-1\})$), where a is the generator of $H^1(S^1, \{-1\})$ ($\cong \mathbb{Z}$) such that $\langle a, [S^1] \rangle = 1$. By [7, II Theorem 7.1], the map $\rho_{M,\partial_1 M}$ is a bijection. If $h : (M, \partial_1 M) \to (S^1, \{-1\})$ is a smooth map such that $\partial_2 h$ ($= h|_{\partial_2 M}$) and h are transversal to 1 in S^1, then submanifold $h^{-1}(1)$ of M has the orientation such that $T(h^{-1}(1)) \oplus \nu(h^{-1}(1), M) = T(M)_{h^{-1}(1)}$ as oriented, real vector bundles. We write $[h^{-1}(1)]$ for the orientation class of $h^{-1}(1)$ in $H_{n-1}(h^{-1}(1))$. Let j_h denote the canonical inclusion $(h^{-1}(1), \emptyset) \to (M, \partial_2 M)$, $\mathrm{pd}_{(M,\partial_1 M)}$ the Poincaré–Lefschetz duality homomorphism $H^1(M, \partial_1 M) \to H_{n-1}(M, \partial_2 M)$, and δ the connecting homomorphism of the Mayer–Vietoris exact sequence

$$\cdots \to H_{n-1}(M) \to H_{n-1}(M, \partial_2 M) \xrightarrow{\delta} H_{n-2}(\partial_2 M) \to \cdots.$$

Speculating the definitions of $\mathrm{pd}_{(M,\partial_1 M)}$ and δ, one obtains the next lemma.

Lemma 5.3. *Let z be an element in $H_{n-1}(M, \partial_2 M)$ and $h : (M, \partial_1 M) \to (S^1, \{-1\})$ a smooth map. If $\partial_2 h$ and h are transversal to 1 in S^1 and $\mathrm{pd}_{(M,\partial_1 M)} \cdot \rho_{M,\partial_1 M}([h]) = z$, then $j_{h*}[h^{-1}(1)] = z$ and $j_{\partial_2 h*}[\partial_2 h^{-1}(1)] = \delta(z)$.*

Since any map is ε-approximated by transversal maps for any $\varepsilon > 0$, one obtains the next lemma.

Lemma 5.4. *Let z be an element in $H_{n-1}(M, \partial_2 M)$ and $g : \partial_2 M \to S^1$ a smooth map. If g is transversal to 1 in S^1 and $j_{g*}[g^{-1}(1)] = \delta(z)$, then there exists a smooth map $h : (M, \partial_1 M) \to (S^1, \{-1\})$ such that h is transversal to 1 in S^1, $j_{h*}\left[h^{-1}(1)\right] = z$ and*

$$h(t, x) = g(x) \quad \left((t, x) \in I \times \partial_2 M = \mathrm{Col}(\partial_2 M, M)\right).$$

Let M be a compact oriented, 2-dimensional, manifold and N a compact, oriented, 1-dimensional, submanifold of $\mathrm{Int}(M)$. Then the normal bundle $\nu(N, M) = N \times \mathbb{R}$ (as oriented, real vector bundles) can be regarded as a tubular neighborhood of N in M. Let $\lambda : \mathbb{R} \to [-2\pi, 2\pi]$ be a smooth, monotone function such that $\lambda(t) = -2\pi$ $(t \leqslant -2)$, $\lambda(t) = t$ $(-1 \leqslant t \leqslant 1)$, and $\lambda(t) = 2\pi$ $(t \geqslant -2)$. Define $h_M^N : M \to S^1$ by

$$h_M^N(x) = \begin{cases} -1 & (\text{if } x \in M \setminus \nu(N, M)), \\ \exp(\lambda(t)\sqrt{-1}) & (x = (y, t) \in N \times \mathbb{R} = \nu(N, M)). \end{cases}$$

Proof of Theorem 5.2. Let $\mathrm{Double}(V)$ denote the double of V. That is

$$\mathrm{Double}(V) = V_1 \bigcup_{\partial V} V_2$$

where V_1 (respectively V_2) is a copy of V with reversed (respectively same) orientation as V. We regard $\mathrm{Im}(\alpha^{(1)}) \subset \mathrm{Int}(V_1)$ and $\mathrm{Im}(\alpha^{(2)}) \subset \mathrm{Int}(V_2)$. Then $\alpha^{(1)}$ becomes orientation reversing. Set

$$\alpha^{(1)'} = \alpha^{(1)} \cdot \coprod_{j \in J(1)} (\mathrm{id}_{S^1} \times \sigma_2) : \coprod_{j \in J(1)} (S_j^1 \times D^2) \to \mathrm{Int}(\mathrm{Double}(V)),$$

$$M = \mathrm{Double}(V) \setminus \mathrm{Int}\left(\mathrm{Im}\left(\alpha^{(1)'} \cup \alpha^{(2)}\right)\right).$$

The assumption $j_{\alpha^{(i)}*}(x) = [\alpha^{(i)}]$ in $H_1(V \setminus \mathrm{Int}(\mathrm{Im}(\alpha^{(i)})))$ $(i = 1, 2)$ implies

$$-\left[\alpha^{(1)'}\right] + \left[\alpha^{(2)}\right] = 0 \quad \text{in } H_1(M).$$

Observing the Mayer–Vietoris exact sequence, we have an element $z \in H_2(M, \partial M)$ such that $\delta(z) = -[\alpha^{(1)'}] + [\alpha^{(2)}]$ in $H_1(\partial M)$. By Lemmas 5.3 and 5.4, there exists a smooth map $h : M \to S^1$ such that h is transversal to 1 in S^1, $j_{h*}([h^{-1}(1)]) = z$, and

$$h(t, x) = h_{\partial M}^N(x) \quad \left((t, x) \in I \times \partial M = \mathrm{Col}(\partial M, M)\right),$$

where $N = \mathrm{Im}((\mathrm{longitude}(\alpha^{(1)'}) \cdot \sigma_2) \cup \mathrm{longitude}(\alpha^{(2)}))$. The oriented manifold $h^{-1}(1)$ is a cobordism between $\mathrm{longitude}(\alpha^{(1)'})$ and $\mathrm{longitude}(\alpha^{(2)})$. Thus, $\alpha^{(1)'}$ and $\alpha^{(2)}$ are $\mathrm{Double}(V)$-cobordant to each other. By Proposition 5.1,

$$\overline{\mu}_{\mathrm{double}(\boldsymbol{\kappa})}\left(\alpha^{(1)'}\right) = \overline{\mu}_{\mathrm{double}(\boldsymbol{\kappa})}\left(\alpha^{(2)}\right).$$

Since

$$\overline{\mu}_{\mathrm{double}(\boldsymbol{\kappa})}\left(\alpha^{(1)'}\right) = \overline{\mu}_{\boldsymbol{\kappa}}\left(\alpha^{(1)}\right),$$

the conclusion of the theorem follows. □

6. Solid torus of genus n

Throughout this section, we regard S^1 and D^2 as the unit circle and the unit disk of $\mathbb{C}$ respectively. Let D_0^3 be a copy of $D^3 \subset \mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, T_i copies of $S^1 \times D^2$ and P_i copies of $I \times D^2$ $(i = 1, \ldots, n)$. Set

$$u_i = \left(\frac{1}{\sqrt{2}}\exp\left(\frac{2\pi i}{n}\sqrt{-1}\right), \frac{1}{\sqrt{2}}\right) \in D_0^3 \quad \text{and}$$

$$v_i = \left(\exp\left(\frac{-\pi}{4}\sqrt{-1}\right), \exp\left(\frac{-\pi}{4}\sqrt{-1}\right)\right) \in T_i \quad (i = 1, \ldots, n).$$

Let U_i (respectively V_i) be tiny closed disk neighborhoods of u_i (respectively v_i) in ∂D_0^3 (respectively ∂T_i). Let $\varphi_i : \{0\} \times D^2 \subset P_i \to U_i$ (respectively $\psi_i : V_i \to \{1\} \times D^2 \subset P_i$) be orientation reversing, linear diffeomorphisms. Now set

$$T(n) = D_0^3 \bigcup_{\coprod_{i=1}^n \varphi_i} \left(\coprod_{i=1}^n P_i\right) \bigcup_{\coprod_{i=1}^n \psi_i} \left(\coprod_{i=1}^n T_i\right)$$

and call $T(n)$ *the solid torus of genus* n. The D_0^3 is called the *central ball* of $T(n)$. For each i, let $e_i = \{1\} \times S^1$ and $f_i = S^1 \times \{1\}$ denote the meridian and longitude of $T_i = S^1 \times D^2$ respectively. Then $H_1(T(n))$ is a free $\mathbb{Z}$-module with basis $\{f_1, \ldots, f_n\}$ and $H_1(\partial T(n))$ is one with basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$. The intersection number of e_i and f_j in $\partial T(n)$ is δ_{ij} (Kronecker's delta).

Definition 6.1. The *algebraic quadratic form* $q_{\partial T(n)}$ *on* $H_1(\partial T(n))$ is defined to be the map

$$H_1(\partial T(n)) \to \mathbb{Z}/2\mathbb{Z}; \quad \sum_{i=1}^n (a_i e_i + b_i f_i) \mapsto \left[\sum_{i=1}^n a_i b_i\right] \quad (a_i, b_i \in \mathbb{Z}).$$

Definition 6.2. A frame (respectively stable frame) $\boldsymbol{\omega}$ over $S^1 \times D^2$ of $T(S^1 \times D^2)$ is said to be *preferable* if $\boldsymbol{\nu}_{S^1 \times D^2} + \boldsymbol{\omega}$ is extensible to a frame (respectively stable frame) over $D^2 \times D^2$ of $T(D^2 \times D^2)$.

By Proposition 2.1, $\boldsymbol{\tau}_p$ is a preferable frame. Thus $\boldsymbol{\omega}$ is a preferable frame (respectively stable frame) if and only if $\boldsymbol{\omega}$ is homotopic (respectively stably regularly homotopic) to $\boldsymbol{\tau}_p$.

Theorem 6.3. *Let* $T(n)$ *be the solid torus of genus* n, z *an element in* $H_1(\partial T(n))$, $\boldsymbol{\kappa}_{(i)}$ $(i = 1, \ldots, n)$ *be preferable frames over* $T_i = S^1 \times D^2$, *respectively, and* $\boldsymbol{\kappa}$ *a frame over* $T(n)$ *of* $T(T(n))$ *such that* $\boldsymbol{\kappa}|_{T_i} = \boldsymbol{\kappa}_{(i)}$. *If*

$$\alpha = \coprod_{i=1}^m \alpha_i \quad (\alpha_i : S^1 \times D^2 \to \mathrm{Int}\,(T(n)))$$

is an orientation preserving, smooth embedding such that $j_{\alpha *}(z) = [\alpha]$ *in* $H_1(T(n) \setminus \mathrm{Int}(\mathrm{Im}(\alpha)))$, *then*

$$q_{\partial T(n)}(z) = \overline{\mu}_{\boldsymbol{\kappa}}(\alpha).$$

Before starting the proof, we prepare several functions which will be needed in the proof. Let $\lambda : \mathbb{R} \to I$ be a smooth, monotone function such that $\lambda(t) = 0$ for any $t \leqslant 0$ and $\lambda(t) = 1$ for any $t \geqslant 1$. For real numbers $p < q$ and a, define $\lambda^a_{p,q} : \mathbb{R} \to \mathbb{R}$ by

$$\lambda^a_{p,q}(t) = 2a\pi\lambda\left(\frac{t-p}{q-p}\right)$$

and, in the case where $0 \leqslant p < q \leqslant 2\pi$ and $a \in \mathbb{Z}$, define $r^a_{p,q} : S^1 \to S^1$ by

$$r^a_{p,q}\left(\exp\left(t\sqrt{-1}\right)\right) = \exp\left(\lambda^a_{p,q}(t)\sqrt{-1}\right) \quad \left(t \in [0, 2\pi)\right).$$

Then the mapping degree of $r^a_{p,q}$ is a.

Proof of Theorem 6.3. Write z in the form $z = \sum_{i=1}^n (a_i e_i + b_i f_i)$ (a_i, $b_i \in \mathbb{Z}$) with respect to the canonical basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ of $H_1(\partial T(n))$ induced by the meridians and longitudes. Take positive integers b_i^+ and b_i^- so that $b_i = b_i^+ - b_i^-$. If there exists an orientation preserving, smooth embedding

$$\alpha' = \coprod_{i\in[1..m]} \coprod_{\varepsilon\in\{\pm\}} \coprod_{j\in[1..b_i^\varepsilon]} \alpha'_{i,\varepsilon,j} \quad \left(\alpha'_{i,\varepsilon,j} : S^1 \times D^2 \to \operatorname{Int}\left(T(n)\right)\right)$$

such that $j_{\alpha'*}(z) = [\alpha']$ and $q_{\partial T(n)}(z) = \overline{\mu}_\kappa(\alpha')$ then by Theorem 5.2, $q_{\partial T(n)}(z) = \overline{\mu}_\kappa(\alpha)$. In the following we show the existence of such α'.

Let $\rho_\varepsilon : \mathbb{C} \to \mathbb{C}$ denote the identity map if $\varepsilon = +$ and σ_2 (the complex conjugation) if $\varepsilon = -$. First define $\beta_{i,\varepsilon,j} : S^1 \times D^2 \to \operatorname{Int}(T(n))$ by

$$\beta_{i,\varepsilon,j}(x, y) = \left(\rho_\varepsilon(x), \frac{\varepsilon \cdot j}{b_i^\varepsilon + 1} + \frac{\varepsilon \cdot \rho_\varepsilon(y)}{3(b_i^\varepsilon + 1)}\right) \in T_i \quad \left((x, y) \in S^1 \times D^2 \subset \mathbb{C} \times \mathbb{C}\right).$$

Set

$$\beta = \coprod_{i,\varepsilon,j} \beta_{i,\varepsilon,j} \quad \left(i \in [1..m],\ \varepsilon \in \{\pm\},\ j \in [1..b_i^\varepsilon]\right) \text{ and } W = T(n) \setminus \operatorname{Int}(\operatorname{Im}(\beta)).$$

Then we obtain

$$\sum_{i,\varepsilon,j} [\beta_{i,\varepsilon,j}] = \sum_i b_i f_i \quad \text{in } H_1(W).$$

Secondly define $\alpha'_{i,\varepsilon,j} : S^1 \times D^2 \to \operatorname{Int}(T(n))$ by

$$\alpha'_{i,\varepsilon,j}(x, y) = \beta_{i,\varepsilon,j}\left(x, \theta^{\varepsilon\cdot a_i}(t) y\right) \quad \left(\theta^{\varepsilon\cdot a_i}(t) = \exp\left(\lambda^{\varepsilon\cdot a_i}_{\pi/2,\pi}(t)\sqrt{-1}\right)\right)$$

($x = \exp(t\sqrt{-1}) \in S^1$ with $t \in [0, 2\pi)$, $y \in D^2$). Set $\alpha' = \coprod_{i,\varepsilon,j} \alpha'_{i,\varepsilon,j}$ ($i \in [1..m]$, $\varepsilon \in \{\pm\}$, $j \in [1..b_i^\varepsilon]$). From $e_i = \sum_{\varepsilon,j} \varepsilon \cdot \operatorname{meridian}(\beta_{i,\varepsilon,j})$ in $H_1(W)$, it follows that $\sum_{i,\varepsilon,j} [\alpha'_{i,\varepsilon,j}] = \sum_i (a_i e_i + b_i f_i)$. Thus $j_{\alpha'*}(z) = [\alpha']$ in $H_1(T(n) \setminus \operatorname{Int}(\operatorname{Im}(\alpha')))$.

It remains to compute $\overline{\mu}_{\boldsymbol{\kappa}}(\alpha')$. Since each $\boldsymbol{\kappa}_i$ is preferable, it is homotopic to the frame $\boldsymbol{\tau}'_p : T_i = S^1 \times D^2 \to P^+(T(T_i))$ defined by

$$\boldsymbol{\tau}'_p(x, y) = \left(\boldsymbol{\tau}_\ell(x), \mathrm{d}L_{g(t)}\boldsymbol{\tau}_s\left(g(t)^{-1}y\right)\right)$$

for $(x,y) \in S^1 \times D^2$ ($x = \exp(t\sqrt{-1})$, $0 \leqslant t < 2\pi$), where $g(t) = \exp(\lambda^1_{0,\pi/2}(t)\sqrt{-1})$. Thus $(\mathrm{d}\alpha'_{i,\varepsilon,j})^*\boldsymbol{\kappa}$ is homotopic to the frame $\boldsymbol{\alpha}'_i : S^1 \times D^2 \to P^+(T(S^1 \times D^2))$ defined by

$$\boldsymbol{\alpha}'_i(x,y) = \big(\boldsymbol{\tau}_\ell(x), \mathrm{d}L_{h(t)}\boldsymbol{\tau}_s\big(h(t)^{-1}y\big)\big)$$

for $(x,y) \in S^1 \times D^2$ ($x = \exp(t\sqrt{-1})$, $0 \leqslant t < 2\pi$), where

$$h(t) = \exp\big(\lambda^{-a_i+1}_{0,2\pi}(t)\sqrt{-1}\big).$$

This implies that $(\mathrm{d}\alpha'_{i,\varepsilon,j})^*\boldsymbol{\kappa}$ is homotopic to $\boldsymbol{\tau}_p$ if and only if

$$-a_i + 1 \equiv 1 \mod 2,$$

which is equivalent to $a_i \equiv 0 \mod 2$. By definition, $\overline{\mu}_{\boldsymbol{\kappa}}(\alpha'_{i,\varepsilon,j}) = [a_i]$ in $\mathbb{Z}/2\mathbb{Z}$ and

$$\overline{\mu}_{\boldsymbol{\kappa}}(\alpha') = \sum_{i,\varepsilon,j}[a_i] = \sum_i [a_i b_i] \quad \text{in } \mathbb{Z}/2\mathbb{Z}. \qquad \square$$

7. Normal map

Let X and Y be compact, connected, oriented, 3-dimensional, manifolds. Let

$$T(\ell_i)_i = D^3_{X,i} \bigcup_{\coprod_{j=1}^{\ell_i}\varphi_{i,j}} \left(\coprod_{j=1}^{\ell_i} P_{i,j}\right) \bigcup_{\coprod_{j=1}^{\ell_i}\psi_{i,j}} \left(\coprod_{j=1}^{\ell_i} T_{i,j}\right)$$

($i \in [1..n]$, $T_{i,j} \cong S^1 \times D^2$) be mutually disjoint, solid tori embedded in $\mathrm{Int}(X)$, such that $\mathrm{genus}(T(\ell_i)_i) = \ell_i$. The $D^3_{X,i}$'s are the central balls of $T(\ell_i)_i$'s, respectively. Let $D^3_{Y,i}$ ($i \in [1..n]$) be mutually disjoint, closed, 3-dimensional balls embedded in $\mathrm{Int}(Y)$. Set

$$U = \bigcup_{i=1}^n T(\ell_i)_i, \qquad V = \bigcup_{i=1}^n D^3_{Y,i}, \qquad X_0 = X \setminus \mathrm{Int}(U), \qquad Y_0 = Y \setminus \mathrm{Int}(V).$$

Let x_i ($i \in [1..n]$) be points of $\partial D^3_{X,i}$ and y_i ($i \in [1..n]$) ones of $\partial D^3_{Y,i}$ respectively. Let $D^2_{X,i}$ be closed, 2-dimensional disks embedded in $\partial D^3_{X,i}$, such that

$$D^2_{X,i} \supset \{x_i\} \cup \bigcup_{j=1}^{\ell_i} \mathrm{Im}(\varphi_{i,j}).$$

Let $\beta_{i,j} : S^1 \times D^2 \to T_{i,j}$ be diffeomorphisms. Let $e_{i,j}$ and $f_{i,j}$ denote the meridians and longitudes of $\beta_{i,j}$, respectively.

In the remainder of the paper, let R be a ring such that $\mathbb{Z} \subseteq R \subseteq \mathbb{Q}$, where $\mathbb{Q}$ is the rational number field. Let Λ denote R or $R[\pi_1(Y)]$ (the group ring of $\pi_1(Y)$ over R). For a map $h : (A,B) \to (A',B')$ of topological pairs, $K_i(h;\Lambda)$ denote the kernel of the homomorphism $h_* : H_i(A,B;\Lambda) \to H_i(A',B';\Lambda)$. If the map h is clear from the context then $K_i(A,B;\Lambda)$ stands for $K_i(h;\Lambda)$.

Maps $f: X \to Y$ in the next context arise naturally in surgery theory.

Definition 7.1. A 1-connected map $f: X \to Y$ such that $f(\partial X) \subset \partial Y$ is called a *prenormal map over* Λ if the following (1)–(5) are satisfied.

(1) $f(D^3_{X,i}) = D^3_{Y,i}$ and $f(D^2_{X,i}) = f(P_{i,j}) = f(T_{i,j}) = \{y_i\}$ ($i \in [1..n]$, $j \in [1..\ell_i]$).
(2) $f(X_0) = Y_0$.
(3) f induces an isomorphism $H_3(X, \partial X; \Lambda) \to H_3(Y, \partial Y; \Lambda)$.
(4) $\partial f = f|_{\partial X}: \partial X \to \partial Y$ induces an isomorphism $H_*(\partial X; \Lambda) \to H_*(\partial Y; \Lambda)$.
(5) $\{f_{i,j} \mid i \in [1..n],\ j \in [1..\ell_i]\}$ generates $K_1(X; \Lambda)$ over Λ.

Remark. Condition (5) is used to obtain $K_1(X_0, \partial U; \Lambda) = 0$.

Given a prenormal map $f: (X, \partial X) \to (Y, \partial Y)$ over R, we obtain the associated butterfly diagram of R-homology groups:

$$\begin{tikzcd}
0 \arrow[dr] \arrow[r, bend left] & K_2(X, X_0; R) = K_2(U, \partial U; R) \arrow[dr] \arrow[rr, bend left] & & K_1(X_0; R) \arrow[dr] \arrow[rr, bend left] & & 0 \\
& K_2(X; R) \arrow[u] \arrow[d] & K_1(\partial U; R) \arrow[ur] \arrow[dr] & & K_1(X; R) \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] \arrow[r, bend right] & K_2(X, U; R) = K_2(X_0, \partial U; R) \arrow[ur] \arrow[rr, bend right] & & K_1(U; R) \arrow[ur] \arrow[rr, bend right] & & 0
\end{tikzcd}$$

similarly to [15, (1) in p. 56]. The algebraic quadratic form $q_{\partial U}: K_1(\partial U; R) \to R/2R$ is given by

$$q_{\partial U}\Bigl(\sum_{i,j} (a_{i,j} e_{i,j} + b_{i,j} f_{i,j})\Bigr) = \sum_{i,j} [a_{i,j} b_{i,j}] \quad \text{in } R/2R \quad (a_{i,j},\ b_{i,j} \in R).$$

This form is trivial if 2 is invertible in R.

The purpose of this paper is to investigate the condition that guarantees

$$q_{\partial U}\bigl(K_2(X_0, \partial U; R)\bigr) = 0,$$

namely the condition that $K_2(X_0, \partial U; R)$ becomes a Lagrangian. For this, bundle data are quite useful as in Wall's surgery theory.

Definition 7.2. A pair (f, b) consisting of a prenormal map over R and an orientation preserving, bundle map $b: T(X) \oplus f^*\eta_- \to \eta_+$ covering f is called a *normal map* over R, where η_+ and η_- are oriented, real vector bundles over Y.

Let (f, b) be a normal map over R as above. Then we can fix frames $\omega_\pm: V \to P^+((\eta_\pm)_V)$ because $(\eta_\pm)_V$ are trivial bundles. Let $b_U{}^*\omega_+$ denote the frame over U of

$T(X) \oplus f^*\eta_-$, induced by $b_U = b| : T(X)_U \oplus (f^*\eta_-)_U \to (\eta_+)_V$ from ω_+, and let $f|_U{}^*\omega_-$ denote the frame over U of $f^*\eta_-$ such that the next diagram commutes

$$\begin{tikzcd} P^+(f|_U{}^*\eta_-) \arrow[r, "\text{canonical}"] & P^+((\eta_-)_V) \\ U \arrow[u, "f|_{U^*}\omega_-"] \arrow[r, "f|_U"] & V \arrow[u, "\omega_-"'] \end{tikzcd} .$$

The next proposition immediately follows from Proposition 2.2.

Proposition 7.3. *There exists a frame $\kappa : U \to P^+(T(U))$ such that $\kappa + f|_U{}^*\omega_-$ is homotopic to $b|_U{}^*\omega_+$. Moreover such a frame is uniquely up to homotopy.*

Hereafter let κ denote a frame over U of $T(U)$ obtained in Proposition 7.3.

Definition 7.4. A smooth immersion $\alpha : S^1 \times D^2 \to U$ is said to be *preferable* if the induced frame $(\mathrm{d}\alpha)^*\kappa$ is preferable (cf. Definition 6.2).

Let $\mathrm{CN}(V)$ be a compact, contractible, manifold neighborhood of V in $\mathrm{Int}(Y)$ ($V \subset \mathrm{Int}(\mathrm{CN}(V))$). Let

$$\alpha^{(i)} = \coprod_{j \in J(i)} \alpha_j^{(i)} \quad \left(i = 1, 2,\ \alpha_j^{(i)} : S^1_{i,j} \times D^2 \to \mathrm{Int}(U), S^1_{i,j} = S^1\right)$$

be disjoint, orientation preserving, smooth embeddings. If there exists a triple (B, β_B, G) consisting of an X-cobordism (B, β_B) between $\alpha^{(1)}$ and $\alpha^{(2)}$ as in Section 5, and a homotopy $G : B \times I \to Y$ satisfying $G(x, 0) = f \cdot \beta_B(x)$ for all $x \in B$, $G(B, 1) \subset \mathrm{CN}(V)$, and $G(x, t) = f \cdot \beta_B(x)$ for all $x \in \beta_B^{-1}(U)$ and $t \in I$, then $\alpha^{(1)}$ and $\alpha^{(2)}$ are said to be $(X, U, \mathrm{CN}(V))$-*cobordant* to each other.

Theorem 7.5. *Let (f, b) be a normal map as in Definition 7.2. If $\alpha^{(1)}$ and $\alpha^{(2)}$ are $(X, U, \mathrm{CN}(V))$-cobordant to each other, then $\overline{\mu}_\kappa(\alpha^{(1)}) = \overline{\mu}_\kappa(\alpha^{(2)})$. In particular, if $\alpha^{(2)}$ is null $(X, U, \mathrm{CN}(V))$-cobordant then $\overline{\mu}_\kappa(\alpha^{(2)}) = 0$.*

Proof. Let (A, β_A, G) be an $(X, U, \mathrm{CN}(V))$-cobordism as above. Set $g = G(-, 1) : B \to \mathrm{CN}(V)$. Since β_B is an immersion, there is a canonical bundle map

$$b_1 = \mathrm{d}\beta_B \oplus \text{canonical} : T(B) \oplus \nu(\beta_B, X) \to T(X).$$

The normal bundle $\nu(\beta_B, X)$ is isomorphic to $\varepsilon_B(\mathbb{R})$. We obtain the bundle map

$$b_2 : T(B) \oplus \varepsilon_B(\mathbb{R}) \oplus (f \cdot \beta_B)^*\eta_- \xrightarrow{b_1 \oplus \text{canonical}} T(X) \oplus f^*\eta_- \xrightarrow{b} \eta_+$$

covering $(f \cdot \beta_B)$. This induces the bundle map

$$b_3 : T(B) \oplus \varepsilon_B(\mathbb{R}) \oplus (f \cdot \beta_B)^*\eta_- \to (f \cdot \beta_B)^*\eta_+$$

covering id_B. There exist bundle maps $b_4 : (f \cdot \beta_B)^*\eta_+ \to g^*\eta_+$ such that $(b_4)_{\beta_B^{-1}(U)} = \mathrm{id}$, and $b_5 : (f \cdot \beta_B)^*\eta_- \to g^*\eta_-$ such that $(b_5)_{\beta_B^{-1}(U)} = \mathrm{id}$. Since the frames $\omega_\pm$ are extensible to frames over $\mathrm{CN}(V)$ of $\eta_\pm$, we regard $\boldsymbol{\omega}_\pm$ as frames over $\mathrm{CN}(V)$. Let $g^*\boldsymbol{\omega}_\pm$ denote the induced frames over B of $g^*\eta_\pm$, $b_4^*g^*\boldsymbol{\omega}_+$ that of $(f\cdot\beta_B)^*\eta_+$, $b_3^*b_4^*g^*\boldsymbol{\omega}_+$ that of $T(B)\oplus\varepsilon_B(\mathbb{R})\ominus(f\cdot\beta_B)^*\eta_-$, $b_5^*g^*\boldsymbol{\omega}_-$ that of $(f\cdot\beta_B)^*\eta_-$. Using $b_3^*b_4^*g^*\boldsymbol{\omega}_+$ and $b_5^*g^*\boldsymbol{\omega}_-$, we obtain a frame $\boldsymbol{\kappa}'$ over B of $T(B)\oplus\varepsilon_B(\mathbb{R})$ such that

$$\boldsymbol{\kappa}'_{\beta_B^{-1}(U)} = (b_1)^*_{\beta_B^{-1}(U)}\boldsymbol{\kappa}.$$

From this fact, $\overline{\mu}(\alpha^{(1)}) = \overline{\mu}(\alpha^{(2)})$ follows similarly to Proposition 5.1. □

8. Main result

The main result of this paper is the next theorem and the present section is devoted to proving it.

Theorem 8.1. *Let (f,b) be a normal map over R, such that f is also a prenormal map over $R[\pi_1(Y)]$. If all the embeddings $\beta_{i,j}: S^1\times D^2\to T_{i,j}\subset U$ ($i\in[1..m]$, $j\in[1..\ell_i]$) are preferable (cf. Definition 7.4) then $q_{\partial U}(\partial(K_2(X_0,\partial U;R)))=0$.*

Proof. If 2 is invertible in R then $q_{\partial U}=0$ and the theorem follows trivially. We suppose that 2 is not invertible in R.

Let $f:(X,\partial X)\to(Y,\partial Y)$ be a prenormal map as in the theorem. Let $p_Y:\widetilde{Y}\to Y$ denote the universal covering of Y, $p_X:\widetilde{X}\to X$ the induced covering from p_Y by f, and $\widetilde{f}:\widetilde{X}\to\widetilde{Y}$ the induced map covering f. Set $\widetilde{U}=p_X^{-1}(U)$, $\widetilde{X}_0=p_X^{-1}(X_0)$, $\widetilde{V}=p_Y^{-1}(V)$, and $\widetilde{Y}_0=p_Y^{-1}(Y_0)$. Since f is a prenormal map also over $R[\pi_1(Y)]$, we obtain the butterfly diagram associated with $\widetilde{f}$:

$$\begin{tikzcd}
0 \arrow[dr] \arrow[rr, bend left] & & K_2(\widetilde{X},\widetilde{X}_0;R)=K_2(\widetilde{U},\partial\widetilde{U};R) \arrow[dr] \arrow[rr, bend left] & & K_1(\widetilde{X}_0;R) \arrow[dr] \arrow[rr, bend left] & & 0 \\
& K_2(\widetilde{X};R) \arrow[ur] \arrow[dr] & & K_1(\partial\widetilde{U};R) \arrow[ur] \arrow[dr] & & K_1(\widetilde{X};R) \arrow[ur] \arrow[dr] & \\
0 \arrow[ur] \arrow[rr, bend right] & & K_2(\widetilde{X},\widetilde{U};R)=K_2(\widetilde{X}_0,\partial\widetilde{U};R) \arrow[ur] \arrow[rr, bend right] & & K_1(\widetilde{U};R) \arrow[ur] \arrow[rr, bend right] & & 0
\end{tikzcd}$$

Moreover, it holds that

$$K_1(\partial U;R)=p_{X*}\big(K_1(\partial\widetilde{U};R)\big),\qquad K_1(X_0;R)=p_{X*}\big(K_1(\widetilde{X}_0;R)\big),$$
$$K_2(X_0,\partial U;R)=p_{X*}\big(K_2(\widetilde{X}_0,\partial\widetilde{U};R)\big).$$

and so on.

Let $\widetilde{z} \in K_2(\widetilde{X}_0, \partial\widetilde{U}; R)$ and set $z = p_{X*}(\widetilde{z}) \in K_2(X_0, \partial U; R)$. We shall show $q_{\partial U}(\partial(z)) = 0$. Since $K_2(\widetilde{X}_0, \partial\widetilde{U}; R) = R \otimes_{\mathbb{Z}} K_2(\widetilde{X}_0, \partial\widetilde{U}; \mathbb{Z})$, there exists an odd integer p such that $p\widetilde{z} \in K_2(\widetilde{X}_0, \partial\widetilde{U}; \mathbb{Z})$. If p is an odd integer then $q_{\partial U}(\partial(pz)) = q_{\partial U}(\partial(z))$. Thus without loss of generality, we can assume that $\widetilde{z} \in K_2(\widetilde{X}_0, \partial\widetilde{U}; \mathbb{Z})$ and hence $z \in K_2(X_0, \partial U; \mathbb{Z})$. Let $\mathrm{CN}(V)$ be a compact, contractible, 3-dimensional submanifold of Y such that $V \subset \mathrm{Int}(\mathrm{CN}(V))$ and $\mathrm{CN}(V) \subset \mathrm{Int}(Y)$.

Lemma 8.2. *Let $f : (X, \partial X) \to (Y, \partial Y)$ and z be as above. Then there exists a triple (A, β_A, G) satisfying the following conditions:*

(1) *A is a compact, oriented, 2-dimensional manifold.*

(2) *$\beta_A : A \to X_0 \setminus \partial X$ is a smooth immersion such that $\beta_A|_{\mathrm{Col}(\partial A, A)}$ is a smooth embedding from $\mathrm{Col}(\partial A, A) = I \times \partial A$ to $\mathrm{Col}(\partial U, X_0) = I \times \partial U$ and*

$$\beta_A|_{\mathrm{Col}(\partial A, A)} = I \times \beta_A|_{\partial A} \quad (\beta_A(\partial A) \subset \partial U).$$

(3) *(A, β_A) represents the element $z \in K_2(X_0, \partial U; \mathbb{Z})$.*

(4) *$G : A \times I \to Y$ is a homotopy relative to ∂A (i.e., $G(x, t) = f \cdot \beta_A(x)$ for all $x \in \partial A$ and $t \in I$) such that $G(x, 0) = f \cdot \beta_A(x)$ for all $x \in A$ and $G(A, 1) \subset \mathrm{CN}(V)$.*

We postpone proving the lemma for a while and continue the proof of Theorem 8.1. Let A, β_A and G be as in the lemma. There exists an orientation preserving smooth embedding $\alpha = \coprod_{i=1}^{m} \alpha_i$ $(\alpha_i : S^1 \times D^2 \to \mathrm{Int}(U))$ such that

$$j_{\alpha*}(\partial(z)) = [\alpha] \quad \text{in } H_1(U \setminus \mathrm{Int}(\mathrm{Im}(\alpha)); \mathbb{Z}).$$

Since $[\partial A] = j_{\alpha*}(\partial(z)) = [\alpha]$ in $H_1(U \setminus \mathrm{Int}(\mathrm{Im}(\alpha)); \mathbb{Z})$, by Lemma 5.4 there exists a smooth map $k : U \setminus \mathrm{Int}(\mathrm{Im}(\alpha)) \to S^1$ such that ∂k and k are transversal to 1 in S^1,

$$k(t, x) = h(x) \quad \big((t, x) \in I \times \partial U = \mathrm{Col}(\partial U, U)\big),$$

and

$$k(t, x) = h^N_{\partial(\mathrm{Im}(\alpha))}(x) \quad \big((t, x) \in I \times \partial\big(\mathrm{Im}(\alpha)\big) = \mathrm{Col}\,\big(\partial\big(\mathrm{Im}(\alpha)\big), U\big)\big),$$

where $N = \bigcup_{i=1}^{m} \mathrm{Im}(\mathrm{longitude}(\alpha_i))$ (for the definition of $h^N_{\partial(\mathrm{Im}(\alpha))}$, see the paragraph before the proof of Theorem 5.2). Set $B = A \cup k^{-1}(1)$ and $\beta_B = \beta_A \cup j_{k^{-1}(1)}$, where $j_{k^{-1}(1)} : k^{-1}(1) \to X$ is the canonical inclusion. Then by the properties described in Lemma 8.2, α is null $(X, U, \mathrm{CN}(V))$-cobordant. By Theorem 7.5, we obtain $\overline{\mu}(\alpha) = 0$.

Proof of Lemma 8.2. The proof is divided into two cases. One is the case where Y is simply connected, and the other is a general case. But the general case follows essentially from the first case.

Case 1 (simply connected). Suppose Y is simply connected. By Lemma 5.3, there exists a smooth map $h : (X_0, \partial X) \to (S^1, \{-1\})$ such that $h|_{\partial U}$ and h are transversal to 1 in S^1 and $j_{h*}[h^{-1}(1)] = z$ in $H_2(X_0, \partial U; \mathbb{Z})$. Set $A' = h^{-1}(1)$ and let $\beta_{A'}$ denote the canonical inclusion of A' to X. Since $A' \cap \partial X = \emptyset$, $\beta_{A'}$ is regarded as a map $(A', \partial A') \to (X_0, \partial U)$. Obtain (A, β_A) by performing 0-dimensional surgery on $(A', \beta_{A'})$ in X_0, such that A is a compact, connected, oriented, 2-dimensional submanifold of $\mathrm{Int}(X_0)$ with $\mathrm{Col}(\partial A, A) = \mathrm{Col}(\partial A', A')$, β_A is the canonical inclusion $(A, \partial A) \to (X_0, \partial U)$, and (A, β_A) represents the element z. Take a smooth triangulation of A such that $\mathrm{Col}(\partial A, A)$ is a subcomplex of it. Let $A^{(1)}$ denote the 1-skeleton of A. Since Y is connected and simply connected, there is a homotopy $G^{(1)} : A \times I \to Y$ relative to ∂A such that $G^{(1)}(-, 0) = f \cdot \beta_A$ and

$$G^{(1)}(A^{(1)} \cup \mathrm{Col}(\partial A, A), 1) \subset \mathrm{CN}(V).$$

Set $g_1 = G^{(1)}(\cdot, 1)$. Note that Y is homotopy equivalent to $Y/\mathrm{CN}(V)$. Thus g_1 is homotopic (relatively to $\mathrm{Col}(\partial A, A)$) to a map $g_2 : A \to Y$ such that $g_2(A) \subset \mathrm{CN}(V)$ if and only if $g_1 : A \to Y/\mathrm{CN}(V)$ is homotopic (relatively to $\mathrm{Col}(\partial A, A)$) to the constant map $A \to \mathrm{CN}(V)/\mathrm{CN}(V) \subset Y/\mathrm{CN}(V)$. Thus Steenrod's obstruction $d(g_1, \mathrm{CN}(V))$ to the existence of a homotopy $G^{(2)} : g_1 \simeq g_2$ lies in $H^2(A, \partial A; \pi_2(Y/\mathrm{CN}(V)))$. Since $Y/\mathrm{CN}(V)$ is simply connected, we can identify $\pi_2(Y/\mathrm{CN}(V))$ with $H_2(Y/\mathrm{CN}(V); \mathbb{Z})$ via Hurewicz' homomorphism. There are also canonical identifications

$$H_2(Y/\mathrm{CN}(V); \mathbb{Z}) = H_2(Y, \mathrm{CN}(V); \mathbb{Z}) = H_2(Y, V; \mathbb{Z}).$$

Let $[A]$ denote the orientation class of A in $H_2(A, \partial A; \mathbb{Z})$. Then

$$H^2(A, \partial A; \pi_2(Y/\mathrm{CN}(V))) = H^2(A, \partial A; H_2(Y, V; \mathbb{Z}))$$

is identified with $H_2(Y, V; \mathbb{Z})$ via the map $x \mapsto \langle x, [A] \rangle$ $(x \in H^2(A, \partial A; H_2(Y, V; \mathbb{Z})))$. By the definition of Steenrod's obstruction, the element $d(g_1, \mathrm{CN}(V)) \in H^2(A, \partial A; H_2(Y, V; \mathbb{Z}))$ corresponds to the element $g_{1*}[A] \in H_2(Y, V; \mathbb{Z}))$ via the identification above. But the latter is equal to $f_*(z)$. Clearly $f_*(z) = 0$ follows from $z \in K_2(X, U; \mathbb{Z})$, hence $d(g_1, \mathrm{CN}(V)) = 0$. There exists a homotopy $G^{(2)} : g_1 \simeq g_2$ relative to $\mathrm{Col}(A, \partial A)$ such that $g_2(A) \subset \mathrm{CN}(V)$. Combining $G^{(1)}$ and $G^{(2)}$, we obtain a desired homotopy $G : f \cdot \beta_A \simeq g_2$.

Case 2 (general). Note that $\widetilde{X}$ is possibly noncompact but $\widetilde{z}$ is obtained in a compact subspace of $\widetilde{X}$. Apply to $\widetilde{z}$ instead of z, the transversality construction employed in Case 1, and obtain a pair $(A, \widetilde{\beta}_A)$ of a compact, connected, oriented, 2-dimensional submanifold A of $\widetilde{X}_0$ and the canonical inclusion $\widetilde{\beta}_A : A \to \widetilde{X}_0$, realizing $\widetilde{z}$. Set $\beta_A = p_X \cdot \widetilde{\beta}_A$. It is easy to control the transversality construction above so that Lemma 8.2(2) is satisfied. Take a compact, contractible, 3-dimensional submanifold $\widetilde{C}$ of Y such that $p_Y(\widetilde{C}) \subset \mathrm{CN}(V)$, $\widetilde{f}(\partial A) \subset \widetilde{C}$, and $(\widetilde{f} \cdot \widetilde{\beta}_A)_*[\widetilde{A}] = 0$ in $H_2(\widetilde{Y}, \widetilde{C}; \mathbb{Z})$. Then by the same argument as in Case 1, there exists a homotopy $\widetilde{G} : A \times I \to \widetilde{Y}$ relative to ∂A such that $\widetilde{G}(x, 0) = \widetilde{f} \cdot \widetilde{\beta}_A(x)$ for all $x \in A$ and $\widetilde{G}(A, 1) \subset \widetilde{C}$. Setting $G = p_Y \cdot \widetilde{G}$, we obtain a desired triple (A, β_A, G). □

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